COSMOLOGICAL PERTURBATIONS

Week 7

OUTLINE

- Gravitational Perturbations
- The Jeans Length
- Newtonian Perturbation Theory
- GR Perturbation Theory

OVER/UNDER DENSITY

- It is simplest to discuss variations from the mean density using a dimensionless parameter, δ.

$$\delta(\vec{r},t) \equiv \frac{\rho(\vec{r},t) - \bar{\rho}(t)}{\bar{\rho}(t)}$$

- Notice that δ ranges from -1 to infinity, it is not symmetric. However, in many cases δ << 1 in which case the asymmetry of the bounds are not important.

- Also δ must be averaged over some scale to have any meaning. The same exact point will have multiple possible values of δ depending on the scale you are smoothing over.

GROWTH OF PERTURBATIONS

- Now consider a spherical region which has a uniform overdensity δ. The gravitational acceleration at the sphere's surface will be

$$\ddot{R} = -\frac{G(\Delta M)}{R^2} = \frac{G}{R^2}\left(\frac{4}{3}\pi R^3 \bar{\rho}\delta\right)$$

or

$$\frac{\ddot{R}}{R} = -\frac{4\pi G\bar{\rho}}{3}\delta(t)$$

- Notice the similarity with the Friedmann equation. A uniform symmetric over/under density behaves like a more or less dense Universe locally.

GROWTH OF PERTURBATIONS

During the collapse (or expansion) of our sphere the mass remains constant.

$$M = \frac{4}{3}\pi\bar{\rho}(1+\delta(t))R(t)^3 \qquad R(t)^3 = R_0[1+\delta(t)]^{-\frac{1}{3}}$$

when δ<<1 $\quad R(t) \approx R_0[1 - \frac{1}{3}\delta(t)]$ $\quad$ so

$$\ddot{R} \approx -\frac{1}{3}R_0\ddot{\delta} \approx -\frac{1}{3}R\ddot{\delta}$$

which when combined with the first equation we got gives

$$\ddot{\delta} = 4\pi G\bar{\rho}\delta$$

GROWTH OF PERTURBATIONS

The solution do this type of equation $\ddot{\delta} = 4\pi G\bar{\rho}\delta$

are exponential functions. $\delta(t) = A_1 e^{t/t_{dyn}} + A_2 e^{-t/t_{dyn}}$

where $t_{dyn} = \frac{1}{\sqrt{4\pi G\bar{\rho}}}$

density fluctuations grow exponentially when small. That being the case, how is it that the initial density fluctuations in the Universe didn't grow to quickly to be large, or for that matter, density fluctuations in the air in this room don't quickly grow to be large?

PRESSURE

The answer is that pressure usually works to counteract gravity. The exponential growth we just calculated is only true if gravity is the only force.

For most physical systems; stars, the atmosphere, the early Universe, pressure counters gravity.

The astrophysicist James Jeans worked out the condition where we expect gravity to be able to win over pressure which is thus called the *Jeans mass* or *Jeans length*.

FLUID DYNAMICS

The two equations that govern fluid dynamics are the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

And Euler's equation, which is just a restatement of Newton's Law

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{\nabla P}{\rho}$$

in this case the only force is pressure

This side is just the fluid acceleration which has two terms because a fluid element is moving. A quantity at that element can change in time and the fluid is now in a different place which the second term accounts for.

INSTABILITY

Lets now consider small perturbations in a uniform fluid. We start by expanding the fluid quantities as

$$\rho = \rho_0 + \rho_1$$

$$P = P_0 + P_1$$

$$v = v_1$$

Putting back into our fluid equation we get

$$\frac{\partial \rho_0}{\partial t} + \frac{\partial \rho_1}{\partial t} + \nabla \cdot ((\rho_0 + \rho_1)\vec{v_1}) = 0$$

$$\frac{\partial \vec{v_1}}{\partial t} + \vec{v_1} \cdot \nabla \vec{v_1} = -\frac{\nabla (P_0 + P_1)}{\rho_0 + \rho_1}$$

INSTABILITY

We can now linearize the equations only keeping first order terms.

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{v_1} = 0 \qquad \rho_0 \frac{\partial \vec{v_1}}{\partial t} = -\nabla P_1$$

Taking a second time derivative of the first equation and the divergence of the second we can add them together and eliminate v_1.

$$\frac{\partial^2 \rho_1}{\partial t^2} + \rho_0 \frac{\partial}{\partial t}(\nabla \cdot \vec{v_1}) = 0 \qquad \nabla \cdot \rho_0 \frac{\partial \vec{v_1}}{\partial t} = -\nabla^2 P_1$$

implies

$$\frac{\partial^2 \rho_1}{\partial t^2} - \nabla^2 P_1 = 0$$

INSTABILITY

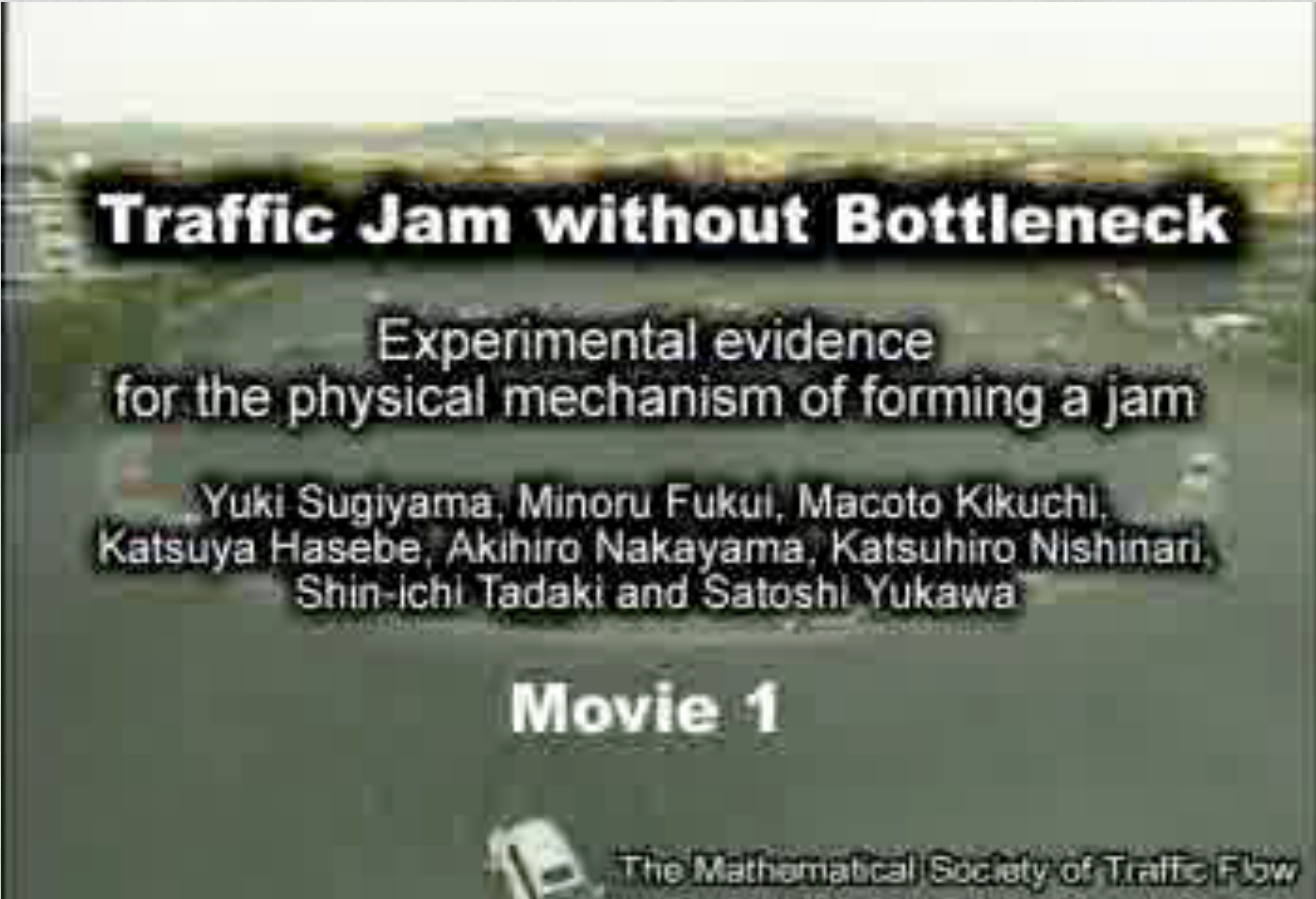
We now have one equation and two variables. To go farther we need an equation of state.

$$P = c_s^2 \rho$$

$$\frac{\partial^2 \rho_1}{\partial t^2} - c_s^2 \nabla^2 \rho_1 = 0$$

Which is just the wave equation and we can now see that c_s is the sound speed in the fluid and the solutions are plane waves.

$\rho_1 \propto e^{i\vec{k}\cdot\vec{x} - i\omega t}$ where $\omega = c_s k$


Traffic Jam without Bottleneck
Experimental evidence
for the physical mechanism of forming a jam
Yuki Sugiyama, Minoru Fukui, Macoto Kikuchi,
Katsuya Hasebe, Akihiro Nakayama, Katsuhiro Nishinari,
Shin-ichi Tadaki and Satoshi Yukawa
Movie 1
The Mathematical Society of Traffic Flow

GRAVITATIONAL INSTABILITIES

Now let us consider the same analysis but with self gravity, in that case we must add Poisson's term to the force.

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{\nabla P}{\rho} - \nabla \phi$$

where $\nabla^2 \phi = 4\pi G \rho$ and we add $\phi = \phi_0 + \phi_1$

To our perturbation analysis. We will take $\nabla\varphi_0=0$ just like we took $\nabla P=0$ earlier. However, formally this doesn't really make sense from Poisson's equation. This is called the *Jeans swindle*, ways to try and get around it are discussed in the literature, like Binney and Tremaine.

GRAVITATIONAL INSTABILITIES

So now keeping only the linear terms we have

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{v_1} = 0$$

$$\rho_0 \frac{\partial \vec{v_1}}{\partial t} = -c_s^2 \nabla \rho_1 - \nabla \phi_1$$

$$\nabla^2 \phi_1 = 4\pi G \rho_1$$

Which can be combined to give

$$\frac{\partial^2 \rho_1}{\partial t^2} + -c_s^2 \nabla^2 \rho_1 = 4\pi G \rho_0 \rho_1$$

This gives the same solutions as before but now $\omega^2 = c_s^2 k^2 - 4\pi G \rho_0$

JEANS LENGTH

Before ω^2 was always positive so the solutions are plane waves, but now ω^2 can be negative which converts our plane waves to exponentially growing or decaying modes.

We've recovered the solution we found before in the absence of pressure. Now we see there is a critical wave number K_J that divides stable and unstable perturbations.

$$K_J \equiv \frac{\sqrt{4\pi G\rho_0}}{c_s}$$

JEANS LENGTH

Turning this into a wavelength we have

$$\lambda_J \equiv c_s \sqrt{\frac{\pi}{G\rho_0}}$$

Perturbations larger than λ_J will collapse , perturbations smaller than λ_J will cause sound waves.

Note that this solution only depends on the size and not the amplitude of the perturbation, as long as the perturbation is small.

JEANS MASS

We can also define a Jeans mass by

$$M_J \equiv \frac{4}{3}\pi\rho_0(\frac{1}{2}\lambda_J)^3 = \frac{c_s^3}{6}\sqrt{\frac{\pi^5}{G^3\rho_0}}$$

For an ideal gas PV=NkT or

$$P = \frac{\rho}{\mu m_H}kT$$ so $$c_s^2 = \frac{kT}{\mu m_H}$$ and

$$M_J = \frac{\pi^{\frac{5}{2}}}{6}\left(\frac{kT}{G\mu m_H}\right)^{\frac{3}{2}}\rho_0^{-\frac{1}{2}}$$

EARTH'S ATMOSPHERE

The Jeans length can also be written as

$$\lambda_J = 2\pi c_s t_{dyn}$$

where t_{dyn} is the dynamical time of the system, $t_{dyn} = (G\rho)^{-1/2}$.

In the Earth's atmosphere c_s~333m/s and t_{dyn} ~ 9hr so the Jeans length is 10^5 km, larger than the planet.

On Earth pressure always wins over self gravity. Only in astrophysics are there systems where gravity wins over pressure.

COSMOLOGICAL

For a flat Universe we have $$H^2 = \frac{8\pi G}{3}\rho$$

which can be written as $$\sqrt{\frac{8}{3}}\pi H^{-1} = \sqrt{\frac{\pi}{G\rho}}$$

So for a single component the density in the Jeans length can be replaced with the Hubble parameter.

$$\lambda_J = c_s\sqrt{\frac{\pi}{G\rho}} = c_s\sqrt{\frac{8}{3}}\pi H^{-1}$$

RADIATION

For radiation $P = \frac{1}{3}\rho c^2$ so $c_s = \sqrt{\frac{1}{3}} c$ and

$$\lambda_J = \pi \sqrt{\frac{8}{9}} \frac{c}{H} \approx 3.0 \frac{c}{H}$$

The Jeans length is 3 times the Hubble distance. In other words perturbations can not grow in a relativistic fluid Universe. Fluctuations only grow when some particle becomes massive enough to no longer be relativistic.

BARYONS

The situation with baryons is somewhat different because until decoupling they are coupled to the radiation. At the time of decoupling the Hubble distance is ~0.2Mpc and the baryon density is $\sim 5.0 \times 10^{-19}$ kg m^{-3}.

So the Jeans mass before decoupling is

$$M_J = \rho_b \left(\frac{4\pi}{3} \lambda_J^3 \right) \approx 7 \times 10^{18} M_\odot$$

This is larger than the largest structures we see today.

BARYONS

Right after decoupling the sound speed drops from $c_s = c/\sqrt{3}$ the sound speed for photons to

$$c_s(\text{baryons}) = \left(\frac{kT}{mc^2}\right)^{\frac{1}{2}} c$$

So the sound speed drops to $c_s \sim 1.5\times10^{-5}c$, and the Jeans mass to $1.5\times10^{5}\ M_\odot$, the mass of the smallest galaxies we see today.

Neither fluctuations of baryons or photons are able to grow before re-ionization, thus the structure we see in the CMB is evidence for a matter component that is not coupled to photons

EXPANDING UNIVERSE

The derivation we did earlier for the growth of perturbations was a static case, but what about in an expanding Universe. We can again consider a point on a slightly over dense sphere. Then

$$\ddot{R} = -\frac{GM}{R^2} = -\frac{4\pi}{3}G\bar{\rho}(1+\delta)R$$

again mass is conserved so

$$M = \frac{4\pi}{3}\bar{\rho}(1+\delta)R(t)^3$$

$$R(t) \propto \bar{\rho}^{-\frac{1}{3}}(1+\delta)^{-\frac{1}{3}}$$

EXPANDING UNIVERSE

But we know $\bar{\rho} \propto a^{-3}$ so we get

$$R(t) \propto a(t)(1+\delta(t))^{-\frac{1}{3}}$$

taking two time derivatives of each side gives

$$\frac{\ddot{R}}{R} = \frac{\ddot{a}}{a} - \frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta}$$

combining with our previous expression gives

$$\frac{\ddot{a}}{a} - \frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} = -\frac{4\pi}{3}G\bar{\rho} - \frac{4\pi}{3}G\bar{\rho}\delta$$

EXPANDING UNIVERSE

But Friedmann's 2nd equation when P=0 is $\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G\bar{\rho}$

so we get $-\frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} = -\frac{4\pi}{3} G\bar{\rho}\delta$ or

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G\bar{\rho}\delta$$

When H=0 this gives the exponential growth we found before. The additional term is a "Hubble" friction, that slows down the growth of perturbations. We can write this equation in terms of Ω_m,

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_M H^2\delta = 0$$

EXPANDING UNIVERSE

When Ω_m is small perturbations do not grow quickly. In the radiation dominated Universe $H = (2t)^{-1}$ and $\Omega_m << 1$ so the perturbation grows as

$$\ddot{\delta} + \frac{1}{t}\dot{\delta} \approx 0 \longrightarrow \delta(t) = B_1 + B_2 \ln t$$

fluctuations can only grow logarithmically with time. In a lambda dominated Universe we have

$$\ddot{\delta} + H_\Lambda \dot{\delta} \approx 0 \longrightarrow \delta(t) = C_1 + C_2 e^{-H_\Lambda t}$$

The fluctuations become constant.
This is what happens during inflation.

EXPANDING UNIVERSE

Only in a matter dominated Universe can fluctuations really grow. If Ω_m=1, then H = 2/(3t) and we have

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0$$

If we guess the solution is a power law At^n, then we get

$$n(n-1)At^{n-2} + \frac{4}{3t}nAt^{n-1} - \frac{2}{3t^2}At^n = 0$$

or $n(n-1) + \frac{4}{3}n - \frac{2}{3} = 0$ which has solutions n=-1 and 2/3.

$\delta(t) = A_1 t^{\frac{2}{3}} + A_2 t^{-1}$ or $\delta \propto t^{\frac{2}{3}} = a(t) = \frac{1}{1+z}$

NEWTONIAN THEORY OF SMALL PERTURBATIONS

NEWTONIAN PERTURBATIONS

Let us know do a proper treatment of the growth of fluctuations, but in the Newtonian limit. We will do the full GR treatment afterwards.

So we consider a fluid with self gravity in a FLRW cosmology.

We will again have the continuity equation, Euler's equation and Poison's equation, but now on our expanding metric.

IDEAL FLUID

$$\frac{D\rho}{Dt} + \rho \nabla_r \cdot \vec{u} = 0$$ continuity

$$\frac{Du}{Dt} = \frac{\nabla_r P}{\rho} - \nabla_r \phi$$ Euler

$$\nabla_r^2 \phi = 4\pi G \rho$$ Poisson

Here r is the proper spatial coordinate. Note that the equations look somewhat similar even though we are including gravity. That is because we have used the convective time derivative,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla_r$$

The convective derivative is for coordinates that move with the fluid. Considering the evolution of a perturbation in a FLRW universe gives

$$\vec{r} = a(t)\vec{x} \qquad \vec{u} = \dot{a}(t)\vec{x} + \vec{v} \qquad \vec{v} \equiv a(t)\dot{\vec{x}}$$

COMOVING COORDINATES

$$\nabla_r \mapsto \frac{1}{a}\nabla_x \qquad\qquad \frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} - \frac{\ddot{a}}{a}\vec{x}\cdot\nabla_x$$

We can now write these equations in comoving coordinates. If we also express density in terms of δ then we get:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla\cdot[(1+\delta)\vec{v}] = 0$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} + \frac{1}{a}(\vec{v}\cdot\nabla)\vec{v} = -\frac{\nabla\Phi}{a} - \frac{\nabla P}{a\bar{\rho}(1+\delta)}$$

$$\nabla^2\Phi = 4\pi G\bar{\rho}a^2\delta \qquad \Phi = \phi + a\ddot{a}x^2/2$$

If there are other smooth energy terms they only change a(t), these equations remain the same. With a cosmological model all we need is an equation of state to solve them.

EQUATION OF STATE

For a non-relativistic monoatomic ideal gas we have

$$P = \frac{\rho}{\mu m_p} k_B T$$

From the first law of thermodynamics

$$d\ln P = \frac{5}{3} d\ln\rho + \frac{2}{3}\frac{\mu m_p}{k_B} S\, d\ln S$$

$$P \propto \rho^{5/3} \exp\left(\frac{2}{3}\frac{\mu m_p}{k_B} S\right)$$

From this we can write

$$\frac{\nabla P}{\bar{\rho}} = c_s^2 \nabla\delta + \frac{2}{3}(1+\delta) T \nabla S$$

where $c_s = \left(\frac{\partial P}{\partial \rho}\right)_S^{1/2}$ is the adiabatic sound speed.

LINEARIZED EQUATIONS

Euler's equation then becomes,

$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} + \frac{1}{a}(\vec{v}\cdot\nabla)\vec{v} = -\frac{\nabla\Phi}{a} - \frac{c_s^2}{a}\nabla\delta - \frac{2\bar{T}}{3a}\nabla S$$

If both v and δ are small we can linearize the equations

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla\cdot\vec{v} = 0$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} = -\frac{\nabla\Phi}{a} - \frac{c_s^2}{a}\nabla\delta - \frac{2\bar{T}}{3a}\nabla S$$

Taking the time derivative of the first equation and combining with the second to get rid of v and Poisson's equation we can get

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} = 4\pi G\bar{\rho}\delta + \frac{c_s^2}{a^2}\nabla^2\delta + \frac{2}{3}\frac{\bar{T}}{a^2}\nabla^2 S$$

ISENTROPIC AND ISOCURVATURE PERTURBATIONS

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} = 4\pi G\bar{\rho}\delta + \frac{c_s^2}{a^2}\nabla^2\delta + \frac{2}{3}\frac{\bar{T}}{a^2}\nabla^2 S$$

We see here that there is a Hubble drag term on the kinematic side.

Perturbations are either in the density (isentropic) or entropy (isocurvature) . Inflation predicts only density (isentropic) perturbations.

Something like variations in the photon-baryon ratio during bariogensis could cause entropy perturbations.

FOURIER TRANSFORM

$$\delta(\vec{x},t) = \sum_k \delta_k(t)\exp(i\vec{k}\cdot\vec{x})$$

If we Fourier transform the equation then ∇ = ik and $\nabla^2 = -k^2$, which make things very simple. Waves evolve independently in the linear regime.

If we have only density perturbations and ignore the expansion of the universe we then get

$$\frac{d^2\delta_k}{dt^2} = -\omega^2\delta_k \qquad \omega^2 = \frac{k^2c_s^2}{a^2} - 4\pi G\bar{\rho} \qquad \lambda_J \equiv \frac{2\pi a}{k_J} = c_s\sqrt{\frac{\pi}{G\bar{\rho}}}$$

We see again that we get a Jeans length tells us when density perturbations will grow.

DIFFUSION DAMPING

- In a multi component Universe any relativistic particles will damp perturbations even if they are able to collapse.
- If the component is the photon-baryon fluid this is called Silk damping.
- This is because relativistic particles will *freestream* out of the over-density. While gravity causes matter to slow down and collapse, photons only lose energy but continue at the same speed, escaping the potential well.
- This process dampens fluctuations, they still grow, but only the matter part contributes to this growth. If the energy density has a sizable contribution from relativistic particles then structure growth will be suppressed.

DIFFUSION DAMPING

The scale on which this damping will be important depends on the typical distance a relativistic particle can travel in a Hubble time. For a photon this is

$$\lambda_d = \sqrt{\frac{ct}{3\sigma_T n_e}}$$

At recombination (z~1100) this corresponds to a Silk damping mass

$$M_d = \frac{\pi}{6}\bar{\rho}_m \lambda_d^3 \approx 2.8 \times 10^{12} (\frac{\Omega_{b,0}}{\Omega_{m.0}})^{-3/2} (\Omega_{m,0} h^2)^{-5/4} M_\odot$$

Calculating this quantity from relativistic perturbation theory gives an answer an order of magnitude larger.

$$M_d \approx 2.7 \times 10^{13} \left(\frac{\Omega_{b,0}}{\Omega_{m.0}}\right)^{-3/2} \left(\frac{X_e}{0.1}\right)^{-3/2} \left(\frac{1+z_{dec}}{1100}\right)^{-15/4} (\Omega_{m,0} h^2)^{-5/4} M_\odot$$

DIFFUSION DAMPING

Notice that this mass scale $\sim 10^{13}\ M_\odot$ is larger then the masses of galaxies.

In a universe where all the matter was atoms structures the sizes of galaxies would not grow before recombination.

However, because most of the matter is dark matter these smaller structures can grow, but just at a reduced rate.

We will see later that the ratio of baryons to matter can be determined by amount of power at these smaller scales in the CMB.

SUMMARY OF PERTURBATION GROWTH

- In a dark energy dominated Universe perturbations do not grow.
- In a radiation dominated Universe growth is suppressed to ln(t).
- In a matter dominated Universe perturbations grow but are suppressed by Hubble drag. They grow as $t^{2/3}$ instead of exponentially.
- In a mixed matter Universe diffusion damping suppresses the growth of perturbations, only the non-relativistic component grows.

ZEL’DOVICH APPROXIMATION

ZEL'DOVICH APPROXIMATION

Since all fluctuations start off extremely small, at latter epochs only the growing modes will have significant amplitude.

$$\delta(\vec{x}, a) = D(a)\delta_i(\vec{x})$$

where δ_i is at some initial time t_i. The growth of the density field is self similar with time. This is true also for the velocity and potential fluctuation. D is called the linear growth factor. The linearized Euler's equation can be integrated for fixed x to give

$$\vec{v} = \frac{\dot{D}}{4\pi G\bar{\rho}_m a^2}\nabla\phi_i(\vec{x}_i)$$

This can then be integrated to get the position as a function of a

$$\vec{x} = \vec{x}_i - \frac{D(a)}{4\pi G\bar{\rho}_m a^3}\nabla\phi_i(\vec{x}_i)$$

ZEL'DOVICH APPROXIMATION

This was worked out by Zel'dovich in 1970. It tells us the displacement and velocity of any point in the linear regime.

The assumption is basically that the initial acceleration a particle would feel simply increases according to the linear growth factor. The particle moves in a straight line with a velocity and displacement that all grow with D.

As long as we are in the linear regime this approximation is pretty good. There is no need to use computational techniques to evolve the density field in this regime.

SEEDING N-BODY SIMULATIONS

- Of course when $\delta \sim 1$ these approximation break down and instead we have to use N-body simulations to model the growth of structure. But linear theory is more accurate then N-body when $\delta \ll 1$ so we don't want to start our N-body simulation till z~50-100.
- To start an N-body simulation first one creates a random density field that matches the power spectrum of the cosmology you want to simulate.
- Then the density field is discretized - replaced with particles. The Zel'dovich approximation is used to move the particles and give them velocities at the redshift where you want to start the simulation.
- It has been shown that some errors occur by using the Zel'dovich approximation as starting conditions. More accurate initial conditions can be determined using 2nd order perturbation theory (2LPT).

N-BODY SIMULATIONS

- The direct N-body calculation goes $O(N^2)$ so can not be done for large number of particles (a million is about the maximum that can be done today).
- In order to get better scaling N-body codes use one of two techniques.
  - Spectral Method - The density field is Fourier transformed, then $\nabla^2 \rightarrow -k^2$ and can be quickly solved.
  - Particle Mesh - At large distances the particles in some region are summed so that only once force calculation is performed.
- In both cases the gravitational potential has to be 'softened' to prevent two body scattering which is unphysical because the particles represent a density field and not actual point masses. So the potential is replaced with

$$\Phi = -G\frac{m_1 m_2}{\sqrt{r^2 + \epsilon^2}}$$


z = 48.4
T = 0.05 Gyr
500 kpc

COSMOLOGICAL PERTURBATION THEORY

with General Relativity

GENERAL FORM

We can right down a equation with all possible types of perturbations to the metric. If we use conformal time ($d\tau = dt/a$) then the unperturbed metric if flat is

$$ds^2 = a^2(\tau)\left(-d\tau^2 + dx^2 + dy^2 + dz^2\right)$$

Perturbations that could be made to the this metric are

$$ds^2 = a^2(\tau)\left(-(1+2A)d\tau^2 - B_i d\tau dx^i + [(1+2D)\delta_{ij} + 2E_{ij}]dx^i dx^j\right)$$

Notice the perturbed metric can have off diagonal elements. Also it is always ok to perturb the flat metric, because the perturbations are small and on small scales the metric is flat.

SCALAR - VECTOR-TENSOR

We see that our 4 perturbation terms are of different types. A and D are scalars, B_i is a vector and D_{ij} is a tensor. It turns out that the different type of perturbations evolve independently so we can consider each type separately.

Each type of perturbation can also be connected to their physical nature. The scale modes are connected to the gravitational potential. The vector modes are connected to gravity-magnetism while the tensor mode is connected to gravitational radiation.

Thus they describe density perturbations, vorticity perturbations and gravitational waves respectively. Inflation predicts there should only be density perturbations and gravitational waves.

CHOICE OF GAUGE

- Recall that we are free to make changes to our coordinate system called a gauge transformation

$$\tilde{x}^0 = x^0 + \delta\tau(x^\mu) \qquad \tilde{x}^i = x^i + \delta x^i(x^\mu)$$

- Gauge transformation will change the form of the scalar perturbations. Thus how we express A and D depend on the gauge. The conformal Newtonian gauge is a common choice. In this gauge the perturbed metric with only density perturbations is is written as

$$ds^2 = a^2(\tau)[-(1+2\Psi)d\tau^2 + (1-2\Phi)\delta_{ij}dx_i dx_j]$$

PERTURBED METRIC

This corresponds to choices of $A_N = \Psi$, $D_N = \Phi$ and $B_N=E_N=0$. Writing this in terms of the metric tensor and switching back to normal time t not conformal time τ gives.

$$g_{00} = -1 - 2\Psi(\vec{x}, t)$$

$$g_{0i} = 0$$

$$g_{ij} = a^2\delta_{ij}(1 + 2\Phi(\vec{x}, t))$$

EINSTEIN'S EQUATION

Now let's solve Einstein's equation for the perturbed metric. For simplicity we will take c=1.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$$

Starting with $T_{\mu\nu}$ it is very hard to solve for the metric, but starting with a metric is is straightforward to solve for $T_{\mu\nu}$. We just need to calculate the Ricci scalar and tensor.

THE CURVATURE TENSOR

Earlier I introduced the curvature tensor without any justification. Now I will give a much better explanation of where it comes from. Our starting point this time will be the covariant derivative and the curvature tensor can be found from considering that the covariant derivative of covariant derivates do not commute.

COVARIANT DERIVATIVE

Remember we introduced the covariant derivative because partial derivatives do not transform correctly in curved space-time. Instead writing the covariant derivative as D_ν we have

$$D_\nu A^\mu \equiv \frac{\partial x'^\alpha}{\partial x^\nu} \frac{\partial x^\mu}{\partial x'^\beta} \frac{\partial A'^\beta}{\partial x'^\alpha}$$

where x' is a coordinate system that is inertial and therefore has the correct transformation behavior. We can define the connection (Christoffel symbol) from the above

$$D_\nu A^\mu = \partial_\nu A^\mu + \Gamma^\mu_{\nu\alpha} A^\alpha \qquad \text{where} \qquad \Gamma^\mu_{\nu\alpha} = \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\alpha}$$

DERIVATIVE OF A TENSOR

The covariant derivative of a tensor is

$$D_\mu(A_\nu B_\lambda) = \partial_\mu(A_\nu B_\lambda) - \Gamma^\alpha_{\mu\nu} A_\alpha B_\lambda - \Gamma^\alpha_{\mu\lambda} A_\nu B_\alpha$$

because the connection is just partial derivatives and so it hast the distributive property. Now lets consider the covariant derivative of the metric.

$$D_\mu(g_{\nu\lambda}) = \partial_\mu(g_{\nu\lambda}) - \Gamma^\alpha_{\mu\nu} g_{\alpha\lambda} - \Gamma^\alpha_{\mu\lambda} g_{\nu\alpha}$$

However there is an inertial frame where the covariant derivative of the metic is zero so it also must be zero in all other frames. Which gives

$$\partial_\mu(g_{\nu\lambda}) = \Gamma^\alpha_{\mu\nu} g_{\alpha\lambda} + \Gamma^\alpha_{\mu\lambda} g_{\nu\alpha}$$

One can then solve for the connection in terms of partials of the metric

$$\Gamma^\nu_{\beta\mu} = \frac{1}{2} g^{\alpha\nu}(\partial_\mu g_{\alpha\beta} + \partial_\beta g_{\alpha\mu} - \partial_\alpha g_{\beta\mu})$$

SO FAR

1. Curved spaces require a covariant derivative
2. A covariant derivative is just a partial derivative plus a connection.
3. The connection can be found by partial derivatives of the metric.

SPACE TIME CURVATURE

Covariant derivatives do not commute and the difference is called the curvature tensor

$$D_\alpha(D_\beta A^\mu) - D_\beta(D_\alpha A^\mu) = R^\mu_{\nu\alpha\beta} A^\nu$$

The covariant derivatives define the curvature tensor in terms of the connection as we've seen before

$$R^\rho{}_{\sigma\mu\nu} = \frac{\partial}{\partial x^\mu}\Gamma^\rho_{\nu\sigma} - \frac{\partial}{\partial x^\nu}\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \qquad R = R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu}$$

PERTURBED METRIC

Now let us return to our perturbed metric. In order to solve Einstein's equations we just need to determine the Ricci tensor for this metric.

$$g_{00} = -1 - 2\Psi(\vec{x}, t)$$

$$g_{0i} = 0$$

$$g_{ij} = a^2 \delta_{ij}(1 + 2\Phi(\vec{x}, t))$$

CHRISTOFFEL SYMBOLS

$$\Gamma^0_{\mu\nu} = \frac{1}{2} g^{0\alpha} [g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}]$$

where $_{,\alpha}$ means the derivative with respect to x^α. The only nonzero component of $g_{0\alpha}$ is the time component which is the inverse of $-1-2\Psi$. To first order that is $-1+2\Psi$ so

$$\Gamma^0_{\mu\nu} = \frac{-1 + 2\Psi}{2} [g_{0\mu,\nu} + g_{0\nu,\mu} - g_{\mu\nu,0}]$$

Let's start with $\mu=0$, $\nu=0$ then all the terms are $g_{00,0} = -2\Psi_{,0}$. Keeping terms only first order in Ψ gives

$$\Gamma^0_{00} = \Psi_{,0}$$

CHRISTOFFEL SYMBOLS

Now let's look at the mixed time and space terms. Then the only nonzero term is $g_{00,i} = 2\Psi_{,i}$

$$\Gamma^0_{0i} = \Gamma^0_{i0} = \Psi_{,i}$$

When both bottom indices are spatial we get

$$\Gamma^0_{ij} = \frac{1-2\Psi}{2}\frac{\partial}{\partial t}[\delta_{ij}a^2(1+2\Phi)]$$

The first order term can be written

$$\Gamma^0_{ij} = \delta_{ij}a^2[H + 2H(\Phi - \Psi) + \Phi_{,0}]$$

FOURIER TRANSFORM

Again it is much easier to work in a Fourier transform of the space than in the real space. Just like we did with the Newtonian petrubations. So our variables (ct,x) go to (ct,k). The main advantage of this is that spatial derivatives can be replaced with ik. So $\Psi_{,i} \rightarrow ik\Psi$.

CHRISTOFFEL SYMBOLS

Then our connections become:

$$\Gamma^0_{0i} = ik_i\Psi$$

$$\Gamma^0_{ij} = \delta_{ij}a^2[H + 2H(\Phi - \Psi) + \Phi_{,0}]$$

$$\Gamma^i_{00} = \frac{ik^i}{a^2}\Psi$$

$$\Gamma^i_{0j} = \delta_{ij}(H + \Phi_{,0})$$

$$\Gamma^i_{jk} = i\Phi[\delta_{ij}k_k + \delta_{ik}k_j - \delta_{jk}k_i]$$

RICCI TENSOR

We still are not done. We have to evaluate the Ricci tensor from the connections. Starting with the R_{00}.

$$R_{00} = \Gamma^{\alpha}_{00,\alpha} - \Gamma^{\alpha}_{0\alpha,0} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{00} - \Gamma^{\alpha}_{\beta 0}\Gamma^{\beta}_{0\alpha}$$

If α=0 the first and second term are the same as are the third and fourth terms, so they all disappear. For other values the first and second terms are just the time derivates of connections we have calculated.

$$\Gamma^{i}_{00,i} = \frac{-k^2}{a^2}\Psi$$

$$-\Gamma^{i}_{0i,0} = -3\left(\frac{\ddot{a}}{a} - H^2 + \Phi,00\right)$$

RICCI TENSOR

$$R_{00} = \Gamma^{\alpha}_{00,\alpha} - \Gamma^{\alpha}_{0\alpha,0} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{00} - \Gamma^{\alpha}_{\beta 0}\Gamma^{\beta}_{0\alpha}$$

For the next term, Γ^{β}_{00} is always 1st order, so only zero order parts of $\Gamma^{i}_{i\beta}$ will give first order terms.

$$\Gamma^{i}_{i\beta}\Gamma^{\beta}_{00} = \Gamma^{i}_{i0}\Gamma^{\beta}_{00} = 3H\Psi_{,0}$$

In the last term if β=0 then both connections are first order and the product is second order. So only spatial terms need be considered.

$$-\Gamma^{i}_{\beta 0}\Gamma^{\beta}_{0i} = -\Gamma^{i}_{j0}\Gamma^{j}_{0i} = -3(H^2 + 2H\Phi_{,0})$$

RICCI TENSOR

So gathering terms the R_{00} piece is

$$R_{00} = -3\frac{\ddot{a}}{a} - \frac{k^2}{a^2}\Psi - 3\Phi_{,00} + 3H(\Psi_{,0} - 2\Phi_{,0})$$

The spatial term is

$$R_{ii} = \left[\left(2a^2H^2 + a\ddot{a}\right)(1 + 2\Phi - 2\Psi) + a^2H(6\Phi_{,0} - \Psi_{,0}) + a^2\Phi_{,00} + k^2\Phi\right]$$

$$R_{ij} = +k_ik_j(\Phi + \Psi)$$

EINSTEIN EQUATION

We are now ready to solve the Einstein equation. We do it for one raised index and one lowered index because that creates some simplification.

$$G^{\mu}_{\nu} = 8\pi G T^{\mu}_{\nu}$$

Starting again with the time-time component.

$$G^0_{\ 0} = g_{00}\left[R_{00} - \frac{1}{2}g_{00}R\right] = (-1+2\Psi)R_{00} - \frac{R}{2}$$

The first order part of G, which we will call δG is then

$$\delta G^0_{\ 0} = -6H\Phi_{,0} + 6\Psi H^2 - 2\frac{k^2\Phi}{a^2}$$

So now we can just equate this with the first order part of $T^0{}_0$

STRESS ENERGY

For matter we have already defined the perturbation

$$\rho_i(1+\delta_i)$$

where the i refers to the species, dark matter or baryons. For radiation we expand in the temperature where θ = δT/T. Since

$$\rho_r \propto T^4 \qquad \frac{\delta\rho_r}{\rho_r} = \frac{4\delta T}{T} = 4\Theta$$

$$T_0^0 = \sum_i \rho_m(1+\delta_m) + \rho_r(1+4\theta_r)$$

Equating the first order parts gives

$$-3H\Phi_{,0} + 3\Psi H^2 - \frac{k^2\Phi}{a^2} = 4\pi G[\rho_d m\delta + \rho_b\delta_b + 4\rho_\gamma\theta_0 + 4\rho_\nu N_0]$$

STRESS ENERGY

Solving all of the equations gives and leaving the stress-energy terms as $\delta T^{\mu}{}_{\nu}$:

$$k^2\Phi + \frac{3\dot{a}}{a}(\dot{\Phi} + \frac{\dot{a}}{a}\Psi) = -4\pi G a^2 |\delta T|_0^0$$

$$k^2(\dot{\Phi} + \frac{\dot{a}}{a}\Psi) = 4\pi G a^2 ik |\delta T|_j^0$$

$$\ddot{\Phi} + \frac{\dot{a}}{a}(\dot{\Psi} + 2\dot{\Phi}) + (\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2})\Psi + \frac{k^2}{3}(\Phi - \Psi) = -\frac{4\pi}{3} G a^2 ik |\delta T|_i^i$$

$$k^2(\Phi - \Psi) = 8\pi G a^2 \frac{3}{2}\left(\frac{k_i k_j}{k^2} - \frac{1}{3}\delta_{ij}\right)\left(T_j^i - \frac{1}{3}\delta_j^i T_j^i\right)$$

These equations combined with an equation of state for the fluid then give the evolution of the perturbations.

LINEAR TRANSFER FUNCTION

THE POWER SPECTRUM

- When we discussed inflation, we introduced the Fourier transforms of the density fluctuations, the power spectrum P(k).
- Inflation suggest $P(k) \sim k^n$, where n is close to 1. This is also called the Harrison-Zeldovich spectrum.
- However, as we have just discussed, these perturbations will not all grow equally. For different length scales or wave numbers k, they grow differently.

TRANSFER FUNCTION

We can define a transfer function to connect the initial perturbations, whatever their sources may be, with the perturbations we see at recombination.

There are different definitions for this function in the literature but it can easily be defined as

$$T(k) = \frac{\Phi(k, t_m)}{A\beta(k)}$$

where $\beta(k)$ is the initial perturbation spectrum, $\Phi(k,t_m)$ is the power spectrum of perturbations after recombination at time t_m and A is a normalization constant to fix the transfer function to 1.0 at some scale.

TRANSFER FUNCTION COMPONENTS

There are two types of effects that the transfer function accounts for:

Damping processes like Silk damping for baryons and free-streaming damping for dark matter.

That sub-horizon perturbations grow differently during radiation and matter dominated eras.

The transfer function thus depends on the contents of the universe.

MATTER RADIATION EQUALITY

- We can start with a Universe of only radiation and dark matter and ignore free streaming. There is the horizon scale of the Universe at matter radiation equality,

$$k_{eq} \equiv \frac{2\pi}{\tau_{eq}} = \frac{2\pi a_{eq}}{ct_{eq}}$$

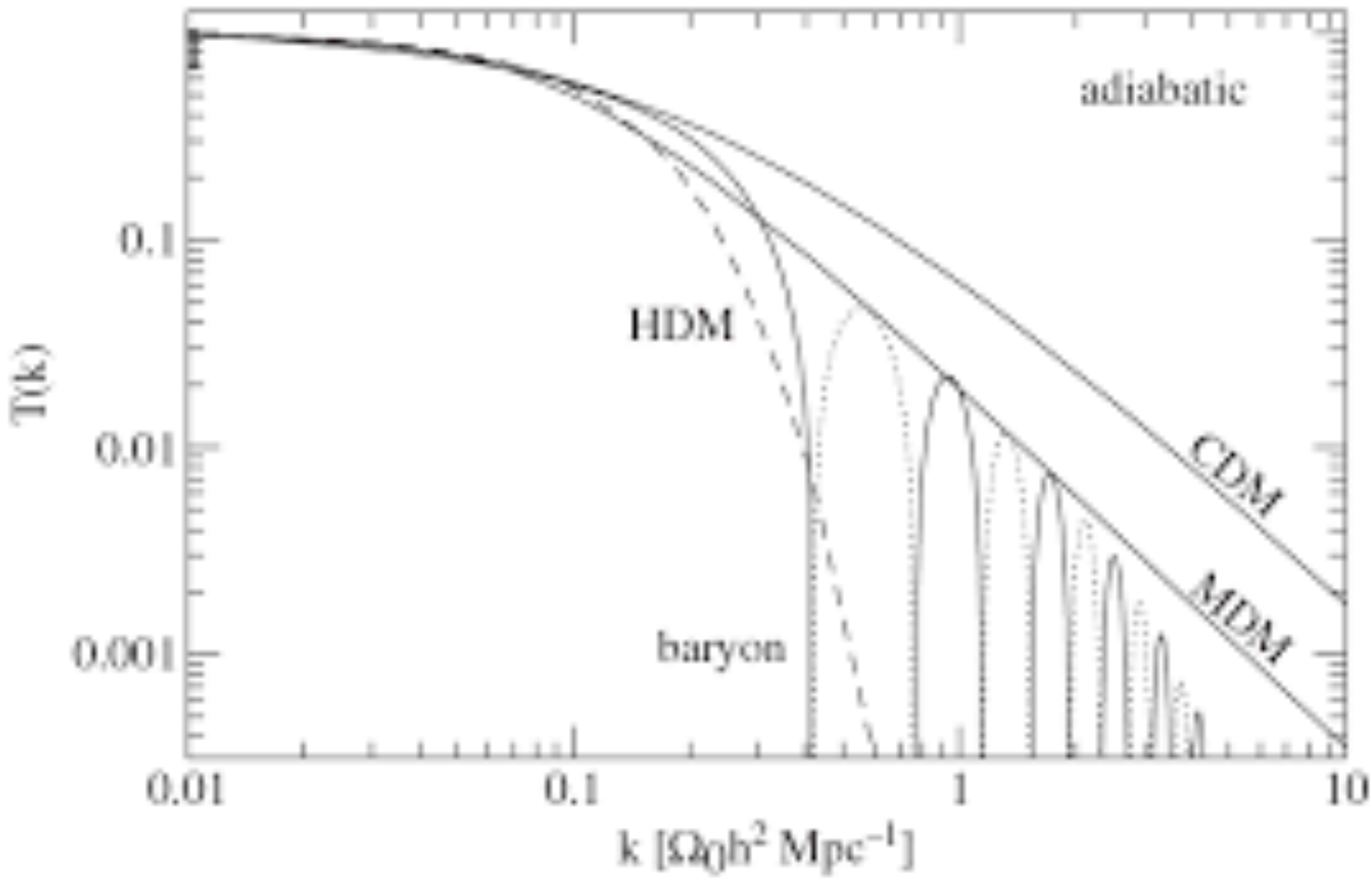
  - $k << k_{eq}$ - enter the horizon when it is matter dominated and have constant amplitude.
  - $k >> k_{eq}$ - density perturbations grow logarithmically during radiation dominance. Isocurvature perturbations do not grow.
  - at t > tm - sub-horizon density perturbations grow as D(t), so Φ grows at D(t)/a(t).

BARYON AND FREESTREAMING

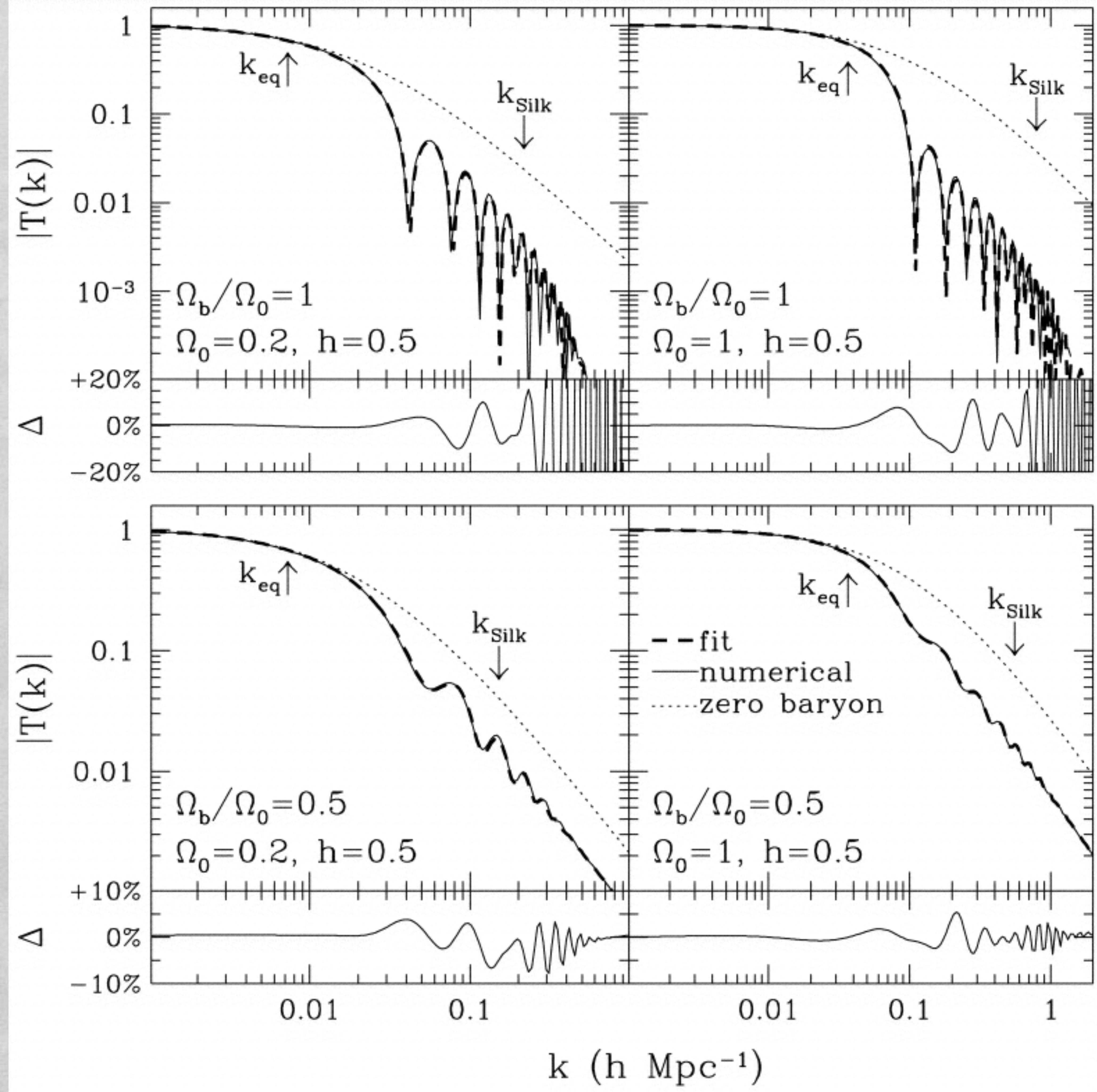
When the dark matter is relativistic it free streams. Thus structures smaller than $k \sim 1/ct_{nr}$ will be suppressed.

The photon-baryon fluid will also free steam (Silk damping). Furthermore, below the Jeans scale the fluid will under go acoustic oscillations.

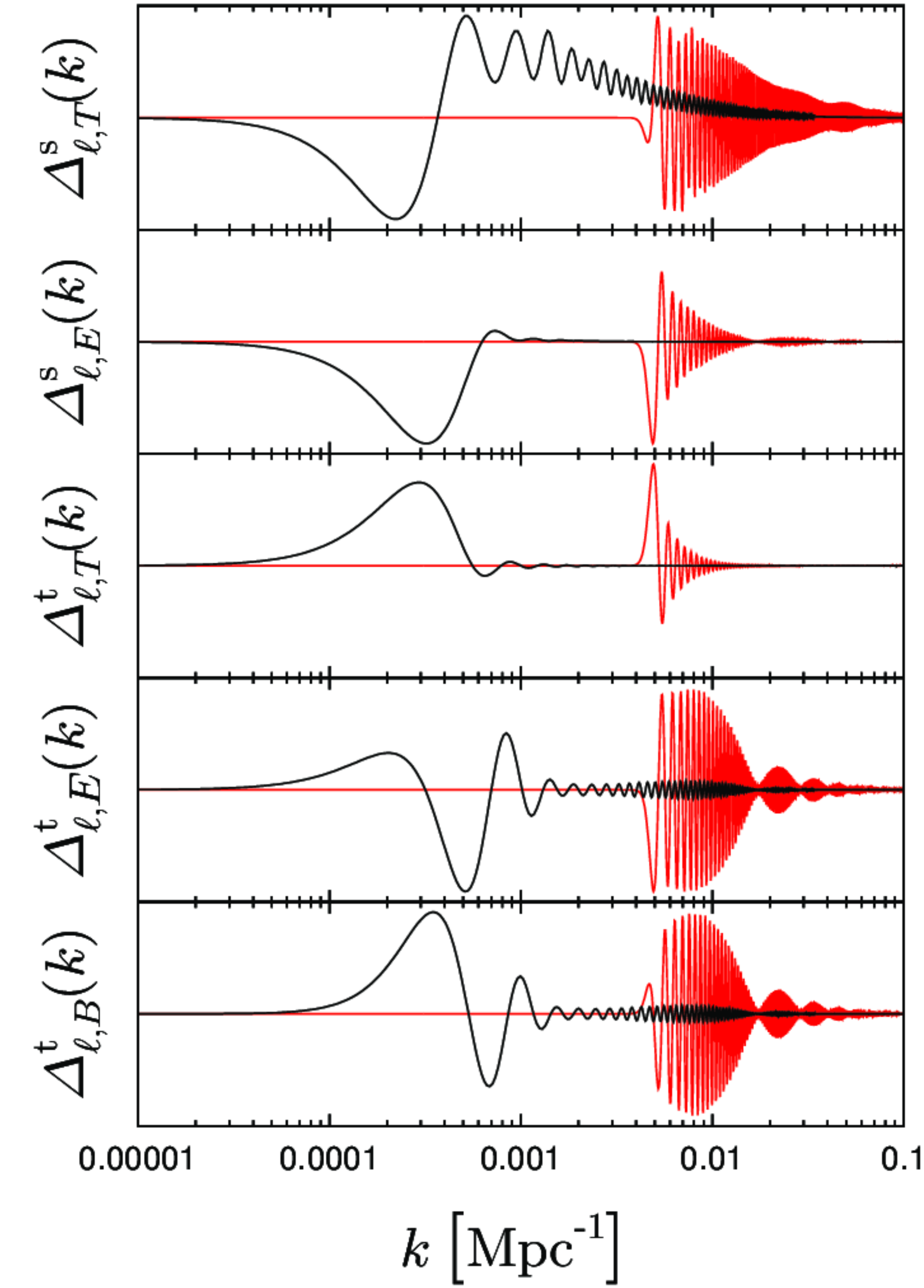
There are public codes (CMBFAST, CAMB, CMBEASY, CLASS) that perform these numerical calculations for standard cosmological models.



Transfer functions for pure baryon model, CDM, HDM and a mixed (30% HDM, 70% CDM) model.



Transfer functions for models with varying matter and baryon ratios.



Scalar temperature fluctuations.

Note that the transfer function depends on the type of perturbation. Previous plots just showed scalar density perturbations.

Tensor modes will grow differently as shown by the bottom three panels.

INITIAL PERTURBATION SPECTRUM

We have taken the initial perturbation spectrum to be

$$P \propto k^n$$

One can instead define a dimensionless quantity $\Delta^2(k)$ which gives the contribution to the power in logarithmic intervals.

$$\Delta^2(k) \equiv \frac{1}{2\pi^2} k^3 P(k) \propto k^{3+n}$$

The corresponding quantity for the gravitational potential is

$$\Delta_\Phi^2(k) \equiv \frac{1}{2\pi^2} k^3 P_\Phi(k) \propto k^{-4} \Delta^2(k) \propto k^{n-1}$$

this is called the Harrison-Zel'dovich spectrum for n=1, because then the potential fluctuations are scale invariant. This is desirable because divergence of the gravitational potential at either small or large scales are not consistent with our Universe.

VARIANCE OF POWER SPECTRUM

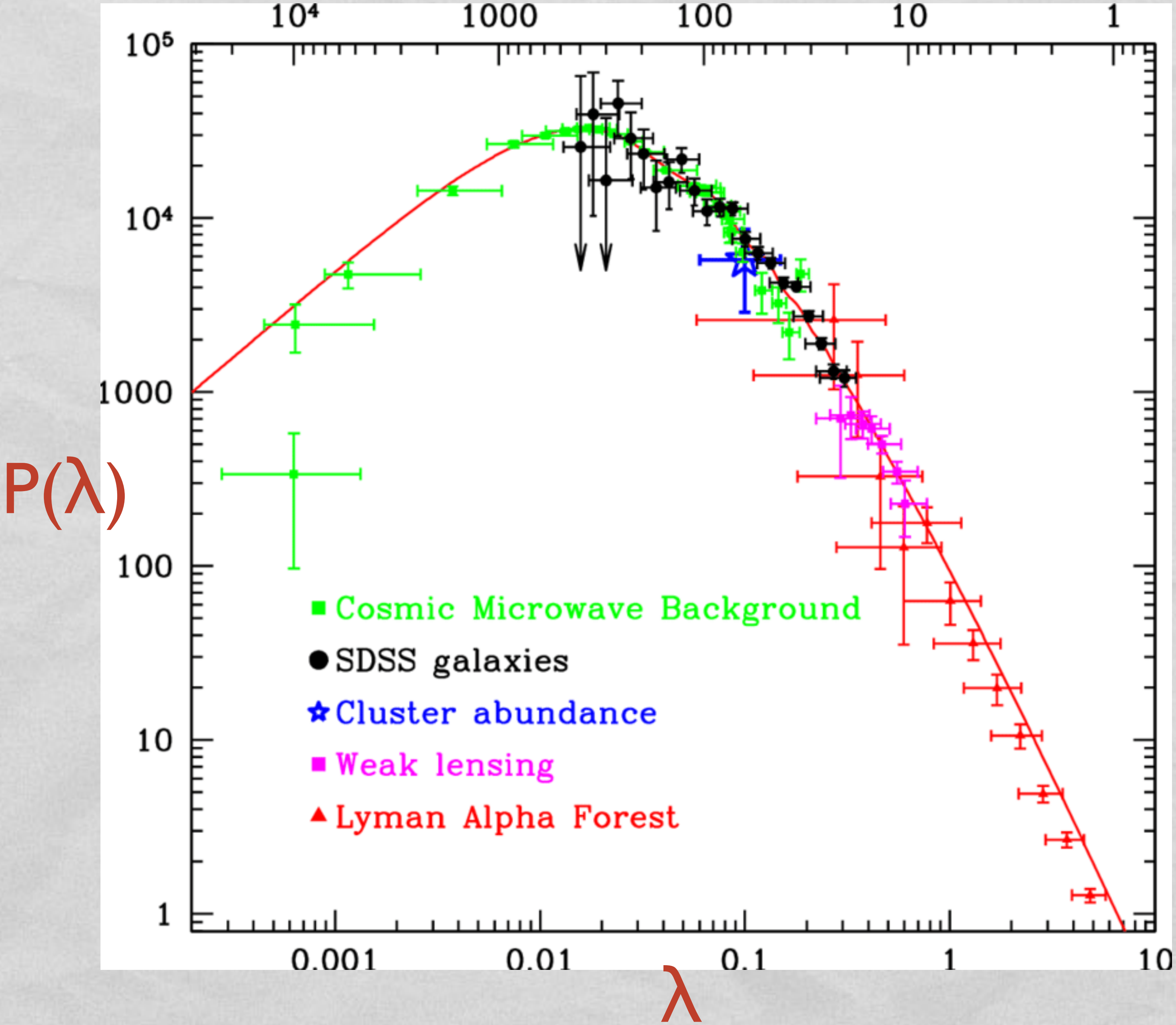
The predicted variance of the density field can be determined from the power spectrum by

$$\sigma^2(R) = \frac{1}{2\pi^2} \int P(k) W_R^2(k) k^2 dk$$

where $W_R(k)$ is a window function often taken to be a top hat.

Since this gives a value around 1.0 for $R=8\ h^{-1}$ Mpc the quantity $\sigma_8 = \sigma(8/h \text{ Mpc})$ is often used as a normalization for the power spectrum.

Note that today scales of 8/h Mpc are nonlinear, but is defined as the linear variance.



In practice the power spectrum is normalized from observations. However, the normalization is still expressed in terms of σ_8 even if that scale is far from where the observations are constrained.

GAUSSIAN PERTURBATIONS

Finally there is the distribution of amplitudes for perturbations. This is usually taken to be Gaussian because without other information this is the best guess and because the statistic of Gaussian processes is well studied.

However there is no reason this has to be true and some models of inflation give nonGaussian perturbations. So looking for evidence of nonGaussian behavior in the fluctuation spectrum is an on going test of the standard ΛCDM model.

PERTURBATIONS SUMMARY

- Perturbations to the metric can be scalar, vector or tensor. Scalar perturbations can be in the density (isoentropic) or the entropy (isocurvature).
- Perturbation growth depends on the dominant energy term:
  - Dark Energy dominated - perturbations don't grow
  - Radiation dominated - perturbations grow logarithmically
  - Matter dominated - perturbations grow as $t^{2/3}$ because of Hubble drag
- Baryons in perturbations can only grow on scales larger than the Jeans length.

PERTURBATIONS SUMMARY

- Free streaming - relativistic particles will free stream out of density perturbations.
  - Silk Damping - when baryons are coupled to photons they will both free stream out of perturbations.
- We guess that the initial perturbation spectrum is k^n, with n~1. Otherwise we have divergence problems at small or large scales (Harrison-Zel'dovich).
- We assume the initial perturbations are Gaussian.
- The amplitude of the initial perturbations are set by observations of the amplitude at later times (CMB).