THE DYNAMICAL UNIVERSE

Week 3

THE METRIC

- So we have seen that the Robertson-Walker metric describes a space-time with the symmetries of the cosmological principle.
- Solving Einstein's equation for this metric gives the Friedman equations that describe how the expansion parameter or scale factor, a(t), depends on the matter-energy properties of the universe.
- First, however, let us just look at a some things that just depend on the metric; co-moving distance, proper distance and geodesics.

COMOVING DISTANCE

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - \kappa r^2/R_0^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]$$

- One thing to notice about the metric is that all time dependance takes place outside the spatial part.
- Two points $(r_1, \theta_1, \varphi_1)$ and $(r_2, \theta_2, \varphi_2)$ will maintain the same coordinate values for all t if there are no forces.
- These are called comoving coordinates and the distance between them comoving distance.


z = 48.4
T = 0.05 Gyr
500 kpc

PROPER DISTANCE

- There are many meanings of distance in cosmology and the end of this section we will go over them.
- However, the most important measure is the proper distance, the distance between 2 points measured at the same time.
- Note that proper distance is not observable. Since we can orient our coordinates as we like we can always make two points only vary by the r coordinate. Then we have.

$$ds^2 = \frac{a^2 dr^2}{1 - \kappa r^2 / R_0^2}$$

PROPER DISTANCE

So the proper distance is give by

$$d_p = a(t)\int_0^r \frac{dr}{\sqrt{1-kr^2}} = \begin{cases} a\sin^{-1}(r) & k=+1 \\ ar & k=0 \\ a\sinh^{-1}(r) & k=-1 \end{cases}$$

If we take the time derivative of the proper distance we get

$$v = \dot{d}_p = \frac{\dot{a}}{a} d_p$$

Which gives us the Hubble Law with

$$H(t) = \frac{\dot{a}}{a}$$

The Hubble Law is a consequence just of the metric. Note that the Hubble 'constant' is not constant but evolves with time.

GEODESIC EQUATION

- One thing to notice about the metric is that all time dependance takes place outside the spatial part.

COSMOLOGICAL REDSHIFT

- Let us consider a wave emitted by some distant galaxy. If massless the wave will follow a geodesic, $ds^2 = 0$. So, $cdt^2 = a^2dr^2$, since I can choose no angular motion.

- Imagine a wave crest emitted at time t_e and observed at time t_o, then

$$c\int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_0^r dr = r$$

- Now consider the next wave crest emitted at $t_e+\lambda_e/c$ and observed at $t_o+\lambda_o/c$. For it

$$c\int_{t_e+\lambda_e/c}^{t_o+\lambda_o/c} \frac{dt}{a(t)} = \int_0^r dr = r$$

COSMOLOGICAL REDSHIFT

- So we see that $$\int_{t_e+\lambda_e/c}^{t_o+\lambda_o/c} \frac{dt}{a(t)} = \int_{t_e}^{t_o} \frac{dt}{a(t)}$$
- If we subtract off the part of the integral that is the same we are left with

$$\int_{t_o}^{t_o+\lambda_o/c} \frac{dt}{a(t)} = \int_{t_e}^{t_e+\lambda_e/c} \frac{dt}{a(t)}$$

- The universe expands very slowly compared to the oscillation of light so we can take a(t) constant and take it out of the integral.

$$\frac{1}{a(t_o)} \int_{t_o}^{t_o+\lambda_o/c} dt = \frac{1}{a(t_e)} \int_{t_e}^{t_e+\lambda_e/c} dt \qquad \frac{\lambda_o}{a(t_o)} = \frac{\lambda_e}{a(t_e)}$$

COSMOLOGICAL REDSHIFT

But we have already encountered this wavelength shift and we call it a redshift, $z=(\lambda_o-\lambda_e)/\lambda_e$.

$$1+z=\frac{\lambda_o}{\lambda_e}=\frac{a(t_o)}{a(t_e)}=\frac{1}{a}$$

So z directly measures a. Note this is a truly cosmological effect even if at low z it acts like a doppler shift. However it is possible to have z > 1 which doesn't make sense as a doppler shift. Some common values of z and a:

z	0	0.5	1	2	3	4	9
a	1	2/3	0.5	1/3	0.25	0.2	0.1

z is observable a is what you have in the metric

DERIVING THE FRIEDMANN EQUATIONS

DERIVING THE FRIEDMANN EQUATIONS

To derive the Friedmann equations we start with the metric and must determine $R_{\mu\nu}$.

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2r^2 & 0 \\ 0 & 0 & 0 & a^2r^2\sin^2\theta \end{bmatrix}$$

$$R_{\mu\nu} = \frac{\partial}{\partial x^\alpha}\Gamma^\alpha_{\nu\mu} - \frac{\partial}{\partial x^\nu}\Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}$$

where $\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\frac{\partial}{\partial x^\mu}g_{\nu\rho} + \frac{\partial}{\partial x^\nu}g_{\rho\mu} - \frac{\partial}{\partial x^\rho}g_{\mu\nu}\right)$

as you can see this is going to require some effort, so I'm just going to do the first term, R_{00}.

$$R_{00} = \frac{\partial}{\partial x^\alpha}\Gamma^\alpha_{00} - \frac{\partial}{\partial x^0}\Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{00} - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha}$$

THE R_{00} TERM

Let's start with $$\Gamma^0_{\mu\nu} = \frac{1}{2} g^{0\beta} \left(\frac{\partial}{\partial x^\mu} g_{\nu\beta} + \frac{\partial}{\partial x^\nu} g_{\beta\mu} - \frac{\partial}{\partial x^\beta} g_{\mu\nu} \right)$$

The metric is diagonal so only $g^{00} \neq 0$. So now we are down to

$$\Gamma^0_{\mu\nu} = \frac{1}{2} g^{00} \left(\frac{\partial}{\partial x^\mu} g_{\nu 0} + \frac{\partial}{\partial x^\nu} g_{0\mu} - \frac{\partial}{\partial x^0} g_{\mu\nu} \right)$$

$g^{00}=g_{00}=-1$. The first two terms are always zero, since only g_{00} is non zero and the derivative of -1 with respect to ct,x,y, or z is 0.

$$\Gamma^0_{\mu\nu} = \frac{1}{2} \frac{\partial}{\partial x^0} g_{\mu\nu} \quad \Gamma^0_{00} = 0 \quad \Gamma^0_{11} = \frac{1}{2} \frac{\partial}{\partial x^0} g_{11} = \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{a^2}{1 - kr^2} \right) = \frac{\dot{a}a}{1 - kr^2}$$

$$\Gamma^0_{22} = \dot{a}ar^2 \qquad \Gamma^0_{33} = \dot{a}ar^2 \sin^2\theta$$

THE R_{00} TERM

Now let's try $$\Gamma^i_{0j} = \frac{1}{2} g^{i\nu} \left(\frac{\partial}{\partial x^j} g_{0\nu} + \frac{\partial}{\partial x^0} g_{j\nu} - \frac{\partial}{\partial x^\nu} g_{0j} \right)$$

Again the metric is diagonal so only if ν=i do we get nonzero terms. This means the first and third term are always zero so we are left with

$$\Gamma^i_{0j} = \frac{1}{2} g^{i\nu} \frac{\partial}{\partial x^0} g_{j\nu}$$

Now you can see we only get nonzero terms for i=j

$$\Gamma^1_{01} = \frac{1}{2} g^{11} \frac{\partial}{\partial x^0} g_{11} = \frac{1}{2} \frac{(1-kr^2)}{a^2} \frac{\partial}{\partial x^0} \left(\frac{a^2}{1-kr^2} \right) = \frac{\dot{a}}{a}$$

$$\Gamma^2_{02} = \frac{1}{2} g^{22} \frac{\partial}{\partial x^0} g_{22} = \frac{1}{2} a^{-2} r^{-2} \frac{\partial}{\partial x^0} (a^2 r^2) = \frac{\dot{a}}{a} \qquad \Gamma^3_{03} = \frac{\dot{a}}{a}$$

THE R_{00} TERM

So the only nonzero Chistoffel symbols are

$$\Gamma^0_{ij} = \frac{\dot{a}}{a} g_{ij} \quad \text{and} \quad \Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j$$

Now let's try

$$R_{00} = \frac{\partial}{\partial x^\alpha}\Gamma^\alpha_{00} - \frac{\partial}{\partial x^0}\Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{00} - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha}$$

The first and third terms are always zero so we have

$$R_{00} = -\frac{\partial}{\partial x^0}\Gamma^i_{0i} - \Gamma^i_{j0}\Gamma^j_{0i} = -3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 - 3\left(\frac{\dot{a}}{a}\right)^2 = -3\frac{\ddot{a}}{a}$$

OTHER TERMS

With lots of hard work you can also find that

$$R_{ij} = -\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2}\right] g_{ij}$$

The Ricci scalar is then

$$R = g^{\mu\nu} R_{\mu\nu} = -3\frac{\ddot{a}}{a} - \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2}\right] g^{ij} g_{ij}$$

$$R = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]$$

EINSTIEN TENSOR

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu}$$

$$G_{00} = -3\frac{\ddot{a}}{a} + 3\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] = \frac{3}{c^2}\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2}\right]$$

where the c^2 factors have reappeared though I lost them somewhere along the way.

EINSTIEN EQUATION

$$G_{00} = \frac{3}{c^2}\frac{\dot{a}^2 + kc^2}{a^2}$$

$$G_{11} = \frac{2a\ddot{a} + \dot{a}^2 + k}{c^2(1 - kr^2)}$$

$$T_{00} = \rho c^2$$

$$T_{11} = \frac{Pa^2}{1 - kr^2}$$

using $$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$ gives

$$\frac{\dot{a}^2 + kc^2}{a^2} = \frac{8\pi G}{3}\rho$$

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + kc^2}{a^2} = -\frac{8\pi G}{c^2}P$$

The G_{22} and G_{33} terms give the same results as the G_{11} term. A consequence of the isotropy we have assumed.

FRIEDMANN EQUATIONS

$$\frac{\dot{a}^2 + kc^2}{a^2} = \frac{8\pi G}{3}\rho \qquad \frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + kc^2}{a^2} = -\frac{8\pi G}{c^2}P$$

These equations are usually rearranged into the following form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \qquad \frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}(\rho c^2 + 3P)$$

This one is also called the acceleration equation

From these equations one can determine the behavior of a(t) knowing ρ and P. However there are 2 equations and 3 unknowns so this is not enough to solve them. One also needs an equation of state, that is a relationship between P and ρ.

FLUID EQUATION

A 3rd useful equation can be derived from the previous two equations called the fluid equation or sometimes the 3rd Friedmann equation.

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - kc^2$$

taking the time derivative of both sides gives

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}\rho 2\dot{a}a - \frac{8\pi G}{3}\dot{\rho}a^2$$

substituting the 2nd Friedmann equation for ä gives

$$2\dot{a}\left[-\frac{4\pi G a}{3c^2}(\rho c^2 + P)\right] = \frac{8\pi G}{3}\rho 2\dot{a}a - \frac{8\pi G}{3}\dot{\rho}a^2$$

or

$$\dot{\rho}c^2 = -3\frac{\dot{a}}{a}(\rho c^2 + P)$$

This is basically a statement of energy conservation, the rate of energy change is equal to the expansion rate times the energy density.

EQUATIONS OF STATE

We basically consider 3 possible equations of state relating pressure to density. We will explore a universe dominated by one of these called a single component universe. Of course we think all 3 relations hold for different types of matter/energy so we really live in a multiple component universe. In general a relation can be written as

$$P = w\rho c^2$$

matter	$P = 0$
radiation (relativistic particles)	$P = \frac{1}{3}\rho c^2$
constant dark energy	$P = -\rho c^2$

SINGLE COMPONENT UNIVERSES

FRIEDMANN EQUATIONS

$$\frac{\dot{a}^2 + kc^2}{a^2} = \frac{8\pi G\rho + \Lambda c^2}{3}$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3}$$

Now with Λ in them. You can either put Λ in explicitly or consider Λ the case of $P=-\rho c^2$.

Let's now explore special cases of just one component.

DENSITY EVOLUTION

We can combine the fluid equation with the equation of state to see how densities will evolve.

$$\dot{\rho}c^2 = -3\frac{\dot{a}}{a}(\rho c^2 + P)$$ and $$P = w\rho c^2$$ give

$$\frac{d\rho}{dt} = -\frac{3}{a}\frac{da}{dt}(1+w)\rho$$ which we can write as

$$\frac{d\rho}{\rho} = -3(1+w)\frac{da}{a}$$

$$\rho(a) = \rho(0)a^{-3(1+w)}$$

So the density of a single component fluid goes to the power of -3(1+w). While we know the universe has different components, when it is dominated by only one then we expect this time of scaling.

MATTER ONLY

In a matter only universe w=0 and k=Λ=0.
From the previous equation we see that

$$\rho_m(a) = \rho_m(0)a^{-3}$$

So in a matter dominated universe density is proportional to 1/ a^3. This makes sense since matter exerts no pressure and is conserved, as a volume gets bigger the density must decrease just enough so that the total matter content remains unchanged.

plugging back into the Friedmann equation we have

$$\left(\frac{\dot{a}}{a}\right)^2 \propto a^{-3} \;\Rightarrow\; da = a^{-\frac{1}{2}}dt \;\Rightarrow\; a^{\frac{3}{2}} \propto t \;\Rightarrow\; a \propto t^{2/3}$$

CRITICAL DENSITY

If we ignore Λ and curvature for a moment the Friedman equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} = H(z)^2$$

This can be used to define a critical density, ρ_c, which is the mean density of the universe with no curvature and no cosmological constant.

$$\rho_c = \frac{3H^2}{8\pi G}$$

Since H_0 is around 70 km/s/Mpc the critical density today is $\rho_c = 9.9\times10^{-27}$ kg/m^3 or 5.9 protons/m^3.

Why that might sound like a pretty low density, best estimates are that the density of normal atoms is less than 1/20 of that or 0.27 protons per cubic meter.

OMEGAS

We can introduce a dimensionless density, by dividing ρ by ρ_c.

$$\Omega_m \equiv \frac{\rho}{\rho_c}$$

Then the first Friedmann equation becomes

$$\Omega_m - \frac{kc^2}{a^2 H^2} + \frac{\Lambda c^2}{3H^2} = 1$$

Defining $\Omega_k \equiv -\frac{kc^2}{a^2 H^2}$ and $\Omega_\Lambda \equiv \frac{\Lambda c^2}{3H^2}$ gives

$$\Omega_m + \Omega_\Lambda = 1 - \Omega_K$$

Remember this is a dynamical equation since all three Ωs depend on H(z).

RADIATION ONLY

How about a universe with radiation only.
Now w=1/3 so we get

$$\rho(a) = \rho(0)a^{-3(1+\frac{1}{3})} = \rho(0)a^{-4}$$

In a radiation (or relativistic particle) dominated universe density goes as $1/a^4$. How can we understand this? Remember the energy of a photon is

$$E = \frac{hc}{\lambda}$$

but in an expanding universe λ goes like a. So E is proportional to 1/a and then you have the $1/a^3$ volume increase to give $1/a^4$.

As the universe expands each photon looses energy and the density of photons decreases.

RADIATION ONLY

plugging back into the Friedmann equation we get

$$\left(\frac{\dot{a}}{a}\right)^2 \propto a^{-4} \quad \Rightarrow \quad a\,da \propto dt \quad \Rightarrow \quad a \propto t^{\frac{1}{2}}$$

A radiation dominated universe grows slower than a matter dominated universe. This at first may seem strange, but the pressure of the radiation contributes to the energy tensor and thus makes gravity stronger compared to matter.

MATTER RADIATION EQUALITY

Note that the matter and radiation density scale and different rates.

$$\rho_m = \frac{\rho_m(0)}{a^3} \qquad \rho_r = \frac{\rho_r(0)}{a^4}$$

That means that as long as there is some relic radiation density today then at some point in the past radiation dominated over matter. And as long as there was some matter originally then at some point it will take over and dominate the expansion.

The time when $\rho_m = \rho_r$ is called matter-radiation equality and can be determined knowing the relative density of each today.

MATTER RADIATION EQUALITY

- As we will see later we can measure the radiation density of the universe from the Cosmic Microwave Background (CMB).
- This is a black body, has a temperature of 2.73K and therefore a energy density of $\rho_{rad} = aT^4 = 7.6\times10^{-15}\,(2.73)^4 = 4.2\times10^{-13}$ ergs/cm^3.
- The current matter density of the universe is around 2.5×10^{-9} ergs/cm^3, or 5950 times greater. We can determine when the two were equal from

$$\rho_{rad} = \rho_{rad,0}(1+z)^4 \qquad \rho_m = \rho_{m,0}(1+z)^3$$

$$\rho_{m,0}(1+z_{eq})^3 = \rho_{rad,0}(1+z_{eq})^4 \quad z_{eq} \simeq \frac{a_0}{a} = \frac{\rho_{m,0}}{1.7\rho_{rad,0}} = 3500$$

where the 1.7 accounts for neutrinos. Since it has been matter domination up to this point we can get the time from

$$t = t_0 a^{-\frac{3}{2}} = 4.8\times10^{-6} t_0$$

The universe has been matter dominated for a long time.

EMPTY UNIVERSE

We can also consider an empty universe, ρ=0. Then

$$\dot{a}^2 = -\frac{\kappa c^2}{R_0^2}$$

One solution to this is $\dot{a}$ =0 and κ=0. An empty, static universe is allowed under Friedmann's equations. It is also possible to have a negatively curved universe (but not a positive).

$$\dot{a} = \pm \frac{c}{R_0}$$

This means the Hubble parameter is a constant so

$$\dot{a} = H_0 a$$

$$\int_0^1 \frac{da}{H_0 a} = \int_0^{t_0} dt$$

The age of the universe is

$$t_0 = H_0^{-1}$$

AGE OF UNIVERSE

We have just seen that the age of the universe is H_0^{-1} if $\Omega_m = \Omega_\Lambda = 0$. On the other hand if $\Omega_K = \Omega_\Lambda = 0$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_m = H_0^2 a^{-3} \quad \Rightarrow \quad da = H_0 a^{-\frac{1}{2}} dt$$

$$H_0^{-1} \int_0^1 a^{\frac{1}{2}} = \int_0^{t_0} dt \quad \Rightarrow \quad t_0 = \frac{2}{3} H_0^{-1}$$

so if Λ=0 the age of the universe is

$$\frac{2}{3} H_0^{-1} \leq t_0 \leq H_0^{-1} \quad \text{or} \quad 9.3\text{Gyr} \leq t_0 \leq 14\text{Gyr}$$

Since the oldest stars are ~12Gyr and we know the universe isn't empty this was a worry and why people brought back the cosmological constant.

LAMBDA UNIVERSE

We can also take w=-1 a dark energy dominated universe. Then

$$\dot{a}^2 = \frac{8\pi G\epsilon_\Lambda}{3c^2}a^2 \quad \text{so} \quad \dot{a} = H_0 a \quad \text{where} \quad H_0 = \sqrt{\frac{8\pi G\epsilon_\Lambda}{3c^2}}$$

and ε_Λ is does not change with a. Thus a grows with time as

$a(t) = e^{H_0(t-t_0)}$ A Lambda universe does not have a big bang.

a approaches zero asymptotically, the universe is infinitely old. Note in this case t_0 is defined by the current time. Most importantly adding Lambda makes the universe older. Unlike a matter universe a increases instead of decreasing with time.

MULTIPLE COMPONENTS

- We know our universe contains radiation and matter. We suspect it has some sort of dark energy and why it seems to be close to flat we can't rule out some curvature.
- Thus we can write a version of the Friedmann equation with the minimal terms we think should be there.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}(\epsilon_r + \epsilon_m + \epsilon_\Lambda) - \frac{kc^2}{a^2}$$

- We can clean this up by replacing the constants with the dimensionless Omegas and $\dot{a}$ with H(z).

$$\left(\frac{H(z)}{H_0}\right)^2 = \Omega_r + \Omega_m + \Omega_\Lambda + \Omega_K$$

MULTIPLE COMPONETS

- We have just seen how all of these components scale with a, so we can replace the Omegas by their current value and how they scale.

$$H^2(z) = H_0^2(\Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{\Lambda,0} + \Omega_{K,0}a^{-2})$$

- This can be written as $H(z) = H_0 E(z)$ where

$$E(z) = \sqrt{\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0} + \Omega_{K,0}(1+z)^2}$$

- In this form we have a formula for the Hubble parameter in terms of observables. This can then be used to derive most cosmological quantities of interest.

MATTER + CURVATURE

Let's consider only matter and curvature for a moment. In this universe we can ask the question, will expansion end? This will happen if H(z) = 0.

$$H^2(z) = \Omega_{m,0} a_{max}^{-3} + \Omega_{K,0} a_{max}^{-2} = 0$$

$$\Omega_{m,0} a_{max}^{-3} = -\Omega_{K,0} a_{max}^{-2} = (\Omega_{m,0} - 1) a_{max}^{2}$$

$$a_{max} = \frac{\Omega_{m,0}}{\Omega_{m,0} - 1}$$

If $\Omega_{m,0} > 1$ then the universe reaches a maximum size and begins to contract after that. Nothing stops the contraction and eventually $a \rightarrow 0$ in what is often called a 'Big Crunch'.

If $\Omega_{m,0} < 1$ then the universe expands forever.

DECELERATION PARAMETER

Since in general the exact functional form of a(t) is not analytic, instead we can do a Taylor expansion for a(t) around the present moment.

$$\frac{a(t)}{a(t_0)} \approx \frac{\dot{a}}{a}(t-t_0) + \frac{1}{2}\frac{\ddot{a}}{a}(t-t_0)^2$$

This can be written

$$a(t) \approx 1 + H_0(t-t_0) - \frac{1}{2}q_0 H_0^2 (t-t_0)^2$$

where $q_0 \equiv -\left(\frac{\ddot{a}a}{\dot{a}^2}\right)_{t=t_0}$ is called the deceleration parameter (notice the sign in the above equation)

This parameterization can be useful because it is physics free, it doesn't even depend on GR. In the standard cosmology it can be expressed as

$$q_0 = \frac{1}{2}\sum_w \Omega_{w,0}(1+3w)$$

show for homework

HOME WORK

- In a positively curved universe containing only matter show that the present age of the universe is

$$t_0 = H_0^{-1} \frac{\Omega_{m,0}}{2(\Omega_{m,0} - 1)^{3/2}} \cos^{-1}\left(\frac{2 - \Omega_{m,0}}{\Omega_{m,0}}\right) - \frac{1}{\Omega_{m,0}}$$

DISTANCES

- In Euclidean space we just have one measure of distance, the metric.
- In cosmology, many of our Euclidean notions of distance turn out to have different values.
- Thus there are different distances with different names; the comoving distance, the transverse comoving distance, the angular diameter distance and the luminosity distance.

HUBBLE VALUES

It can be useful to define a time and length based on the value of the Hubble parameter today. These are called the Hubble time t_H and the Hubble distance D_H.

$$t_H = \frac{1}{H_0} \qquad D_H = \frac{c}{H_0}$$

If H_0 = 70 km/s/Mpc then t_H = 14 Gyr and D_H = 4283 Mpc. These numbers give you roughly the age and observable size of the universe, though they are modified by factors of order unity depending on the exact cosmology.

PROPER DISTANCE

As mentioned before proper distance is the distance between two points at the same time.

$$d_p = a(t)\int_0^r \frac{dr}{\sqrt{1-kr^2}} = \begin{cases} a\sin^{-1}(r) & k=+1 \\ ar & k=0 \\ a\sinh^{-1}(r) & k=-1 \end{cases}$$

The proper distance of some light emitted at time t_e that we observe at t_o is

$$d_p = c\int_{t_o}^{t_e} \frac{dt}{a(t)}$$

COMOVING DISTANCE

The comoving distance is the distance between points that remains fixed if the points are moving with the Hubble flow (i.e. not moving with respect to the metric).

$$d_c(z) = (1+z)d_p$$

$$d_c(z) = D_H \int_0^z \frac{d_{z'}}{E(z')}$$

Note that the proper distance between two points does change with a, while the comoving distance does not. The comoving distance is in some sense fundamental because all other distances can be derived from it.

PROPER MOTION DISTANCE

The transverse comoving distance or proper motion distance is the distance between points at the same redshift but separated by an angle $\delta\theta$, where the distance between the points is then $d_M\delta\theta$. It is given by

$$d_m(z) = \begin{cases} \frac{D_h}{\sqrt{\Omega_K}} \sinh\left(\sqrt{\Omega_k}\frac{d_c}{D_H}\right) & \Omega_k > 0 \\ d_c & \Omega_k = 0 \\ \frac{D_h}{\sqrt{\Omega_K}} \sin\left(\sqrt{|\Omega_k|}\frac{d_c}{D_H}\right) & \Omega_k < 0 \end{cases}$$

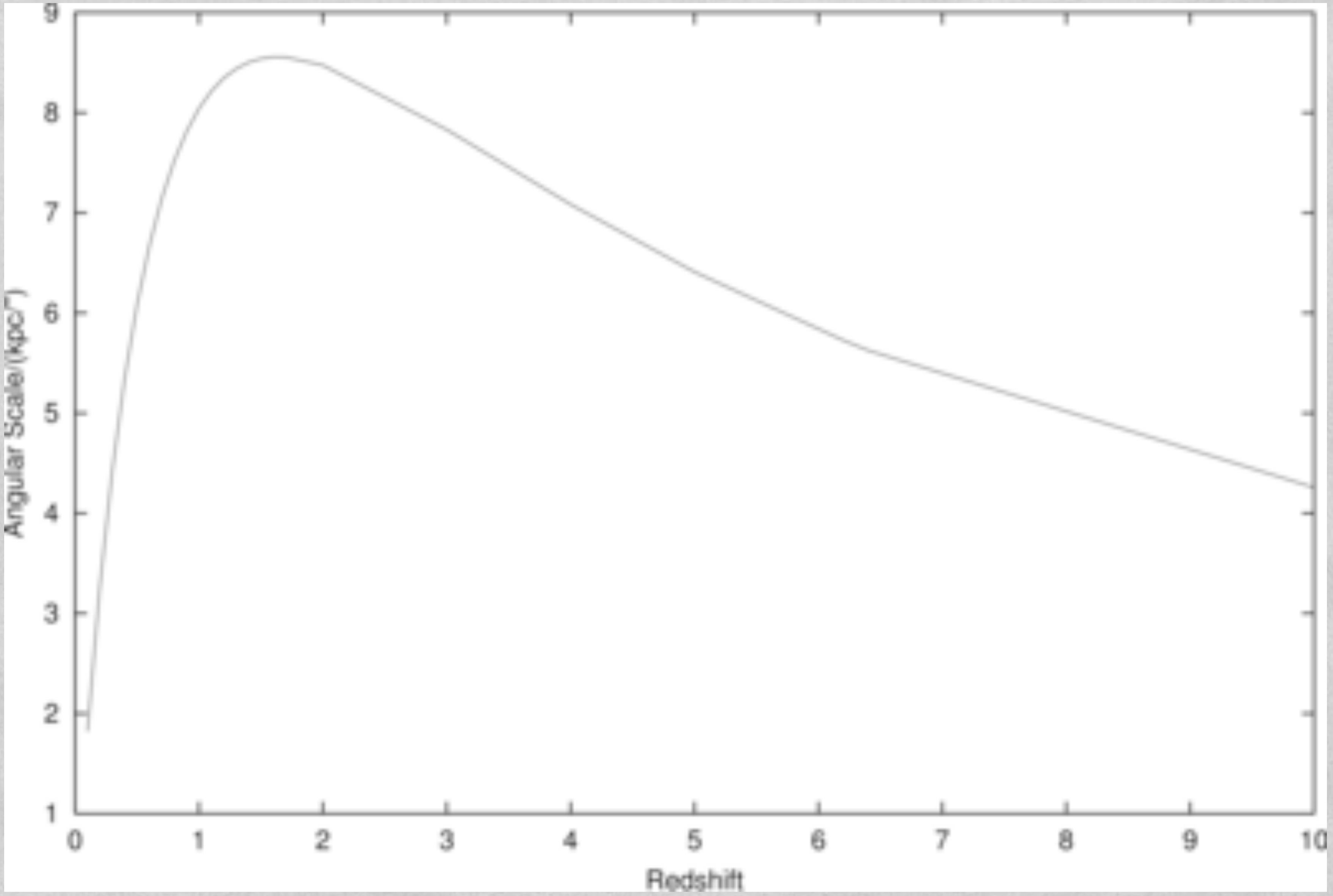
This distance is also the ratio of the actual transverse velocity (length/time) of an object to its proper motion (radians/time).

ANGULAR DIAMETER DISTANCE

Angular diameter distance is defined as the ratio of an objects physical transverse size to its angular size.

$$d_a(z) = \frac{d_m(z)}{(1+z)}$$

Note that, depending on cosmology, objects may not get smaller as they get farther away. This is the case for the favored cosmology.



LUMINOSITY DISTANCE

Luminosity distance is what you would use to get the flux from a light source.

$$f = \frac{L}{4\pi d_L^2} \qquad d_L(z) = (1+z)d_m = (1+z)^2 d_a$$

The later equality can be understood as the surface brightness of a receding object is reduced by a factor $(1+z)^{-4}$, and the angular area goes down as d_a^{-2}.

COMOVING VOLUME

Finally the comoving volume is the volume measure in which number densities of unchanging objects remain the same.

$$dV_c = D_H \frac{(1+z)^2 d_a^2}{E(z)} d\Omega dz$$

The probability of a line of sight intersecting an object with comoving density n(z) and cross section σ(z) is

$$dP = n(z)\sigma(z) D_H \frac{(1+z)^2}{E(z)} dz$$