# GENERAL RELATIVITY AND THE ROBERTSON-WALKER METRIC

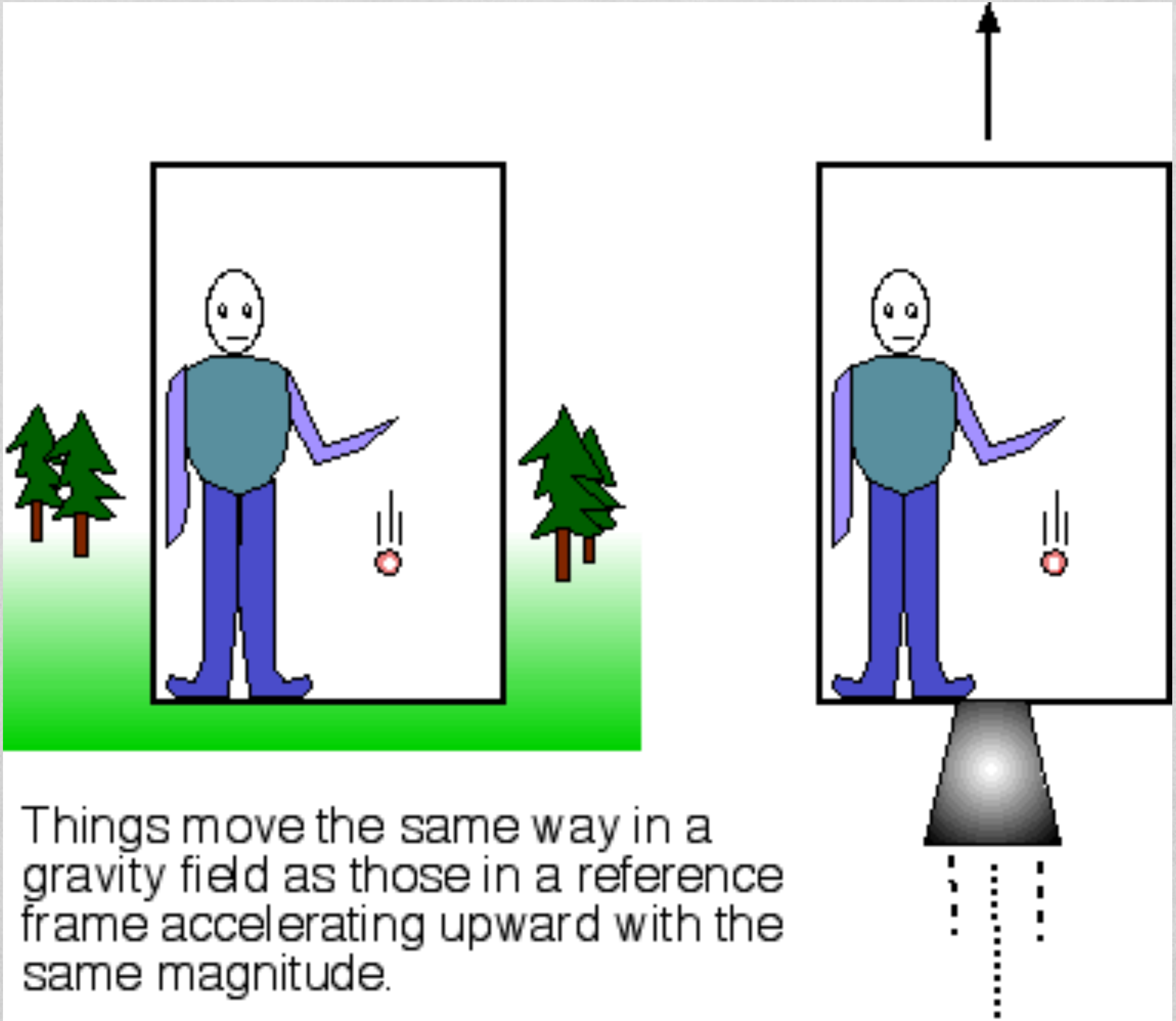
Week 2

# EQUIVALENCE PRINCIPLE

- One of the remarkable facts of Newtonian dynamics is that the gravitational charge is equal to the inertial mass of an object.
- In the equations $F_g = G\frac{Mm}{r^2}$ and $F = ma$ the m is the same.
- This leads to the surprising result that two objects fall at the same speed on the Earth, or that the orbits of planets are independent of the mass of the planet.
- If you think about it, it is quite strange that this is the case, it isn't true for any other force.

# GRAVITY OR ACCELERATION

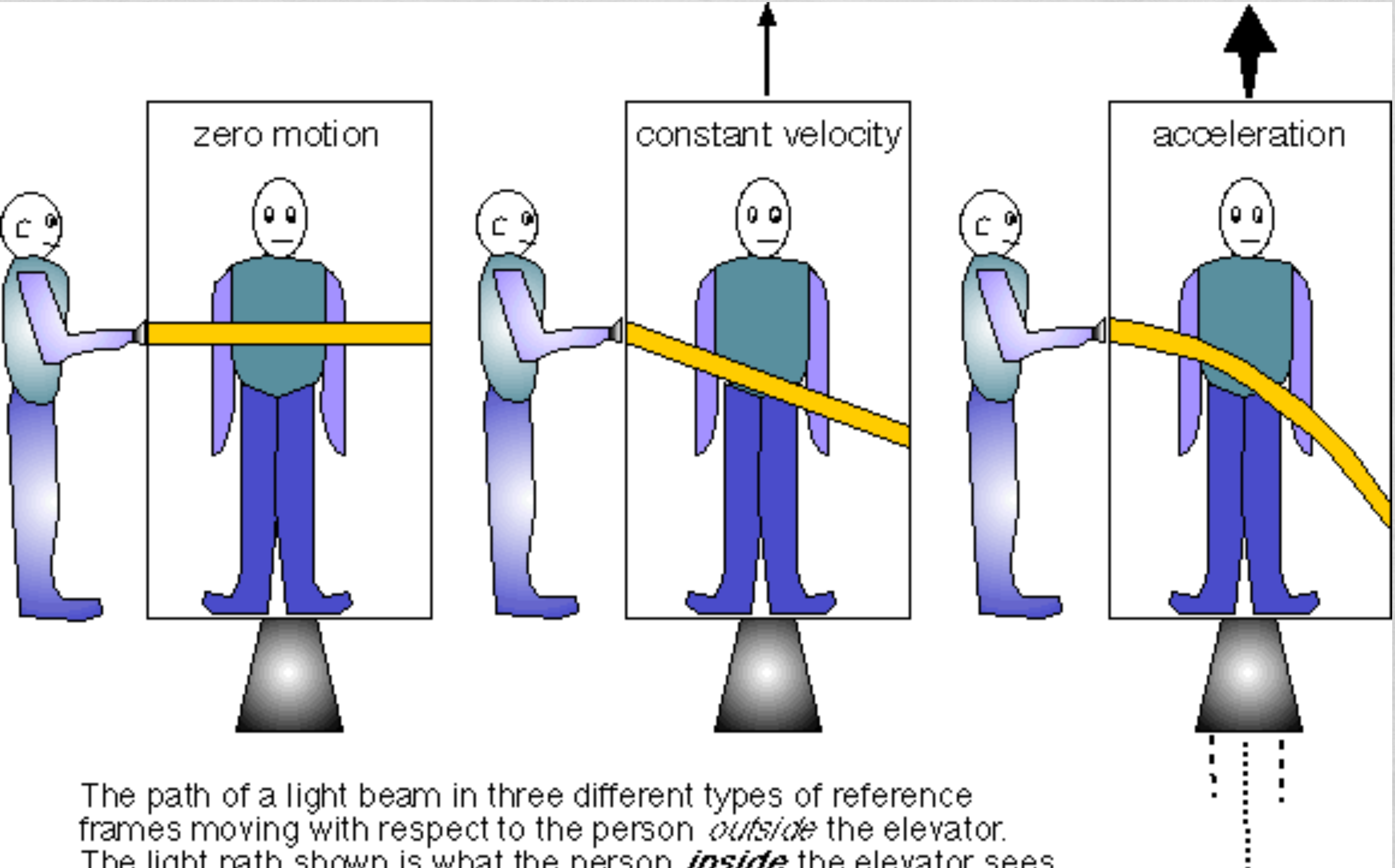
- General relativity is based on the idea that gravity and acceleration are indistinguishable.
- If you are in an enclosed box on Earth, or in a box on a spaceship accelerating at $9.81 ms^{-2}$, things feel exactly the same and the laws of physics must be the same.
- Galileo’s experiment would give you exactly the same answer in both cases. Projectile motion would be the same. All of Newton’s Laws would work exactly the same, locally.



Things move the same way in a gravity field as those in a reference frame accelerating upward with the same magnitude.

# BENDING LIGHT

- Now lets try to examine something new. What would happen to a laser beam perpendicular to the gravity/acceleration.
- In an accelerating spaceship the light would start on one side of the spaceship, but during the time it took to cross the spaceship the ship would move a little so the light would hit the other side and a slightly lower point.
- The lights motion would seem to be curved.




The path of a light beam in three different types of reference frames moving with respect to the person *outside* the elevator. The light path shown is what the person ***inside*** the elevator sees. Under large acceleration, the beam of light will curve downward. It should also do that in a region of strong gravity.

# GRAVITY MUST CURVE SPACE-TIME

- So gravity must curve light.
- But from special relativity we know the fastest you can travel between two points (and thus the shortest distance) is the speed of light.
- The only way a curve can be the shortest distance between 2 points is if the space is not flat.
- Gravity must curve space-time.

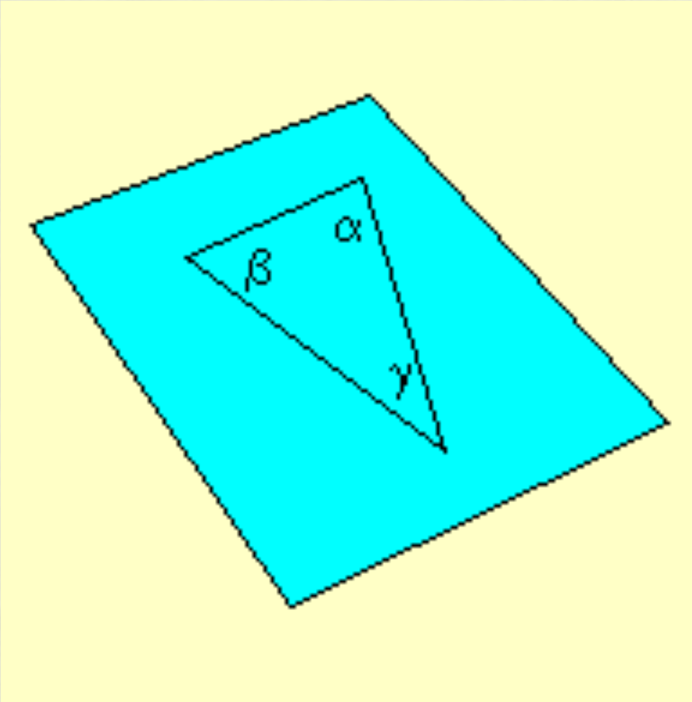
# CURVATURE ON A 2D SURFACE

# TWO DIMENSIONAL SURFACES

- Let's first consider 2D spaces because they are easy to visualize.

- The space you are used to is Euclidean space, an infinite flat plan (x,y). The rules of Euclidean geometry are valid in this space; parallel lines never meet, the sum of the angles in a triangle are 180, etc.

$$\alpha + \beta + \gamma = 180$$




- In this space the distance between two is given by the Pythagorean Theorem

$$ds^2 = dx^2 + dy^2$$

- notice we consider only a small distance, on this space it is true also at large distances, but that won't be the case in other spaces.

# METRIC SPACES

$$ds^2 = dx^2 + dy^2$$

- This equation is called a metric. A space that has a metric that meets the following conditions is called a metric space.
  1. d(x,y) > 0 (positive)
  2. d(x,x) = 0 (identity)
  3. d(x,y)=d(y,x) (symmetry)
  4. d(x,z) ≤ d(x,y) + d(y,z) (triangle inequality)
- Under a coordinate transformation the metric can take a different form, but it is the same metric.
- For example, in polar coordinates the metric on a plane takes the form:

$$ds^2 = dr^2 + r^2 d\theta^2$$

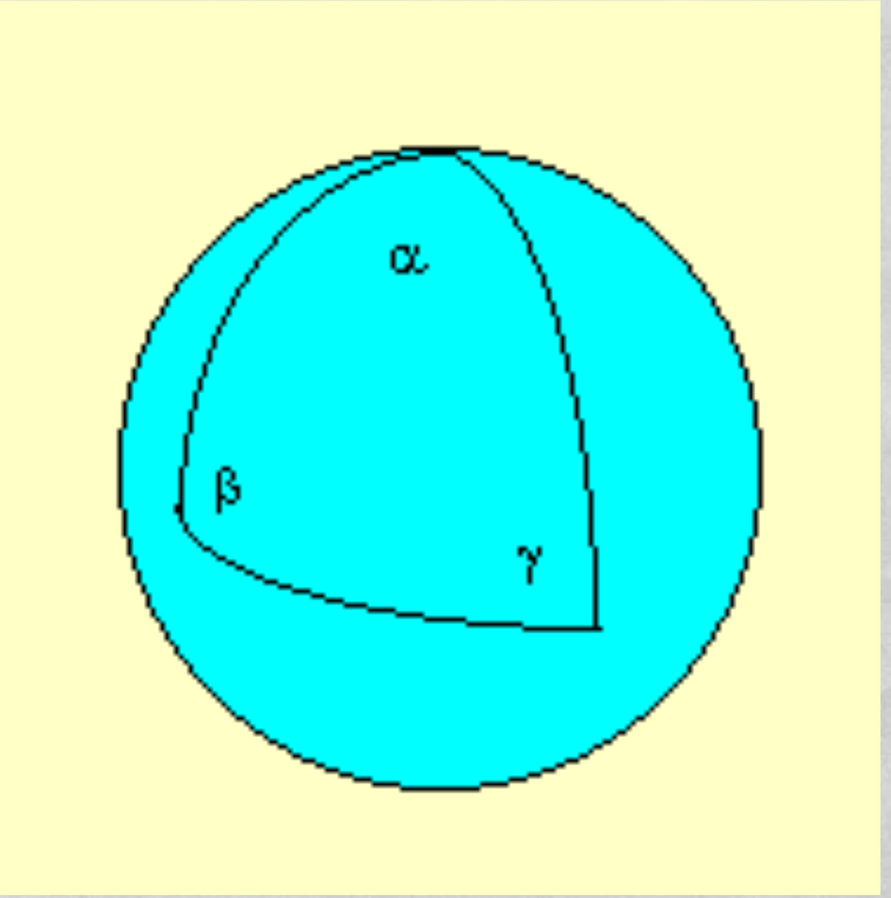
# POSITIVE CURVATURE

- The simplest non flat space we can think of is the surface of a sphere. This is called positive curvature.
- On this surface Euclidean geometry does not hold, parallel lines do meet and the angles of a triangle sum to:

$$\alpha + \beta + \gamma = 180 + \frac{A}{R^2}$$

- where A is the area of the triangle and R is radius of the sphere. The metric for this surface is:

$$ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\theta^2$$



# NEGATIVE CURVATURE

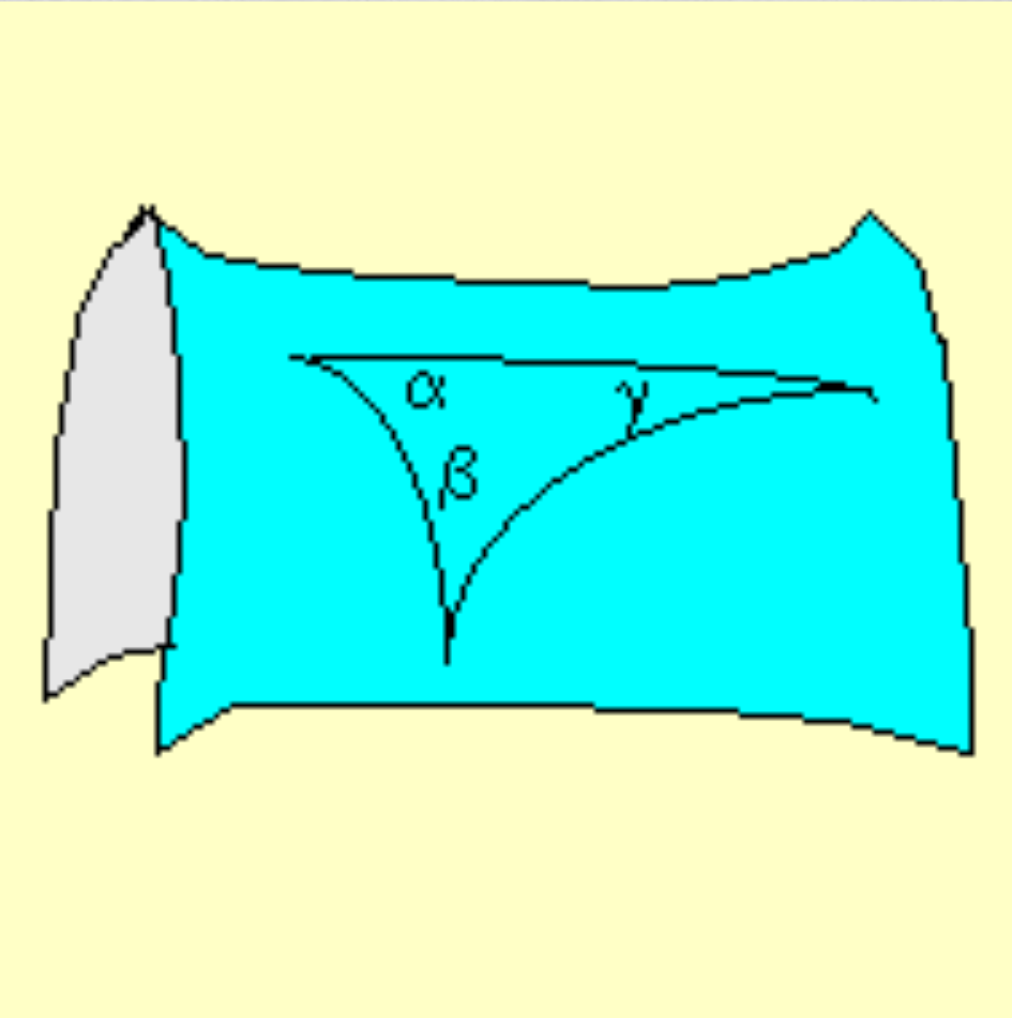
- Another possibility is a space with negative curvature. While this is harder to visualize a saddle has constant negative curvature in the center.

- Again Euclidean geometry does not hold, the angles of a triangle sum to:

$$\alpha + \beta + \gamma = 180 - \frac{A}{R^2}$$

- The metric of this space is:

$$ds^2 = dr^2 + R^2 \sinh^2\left(\frac{r}{R}\right) d\theta^2$$



# 3D SPACES

We can generalize these metrics to three dimensional spaces. Then we have for flat, positive and negative curvature:

$$ds^2 = dr^2 + S_\kappa^2(r)(d\theta^2 + \sin^2\theta d\phi^2)$$

$$S_\kappa(r) = \begin{cases} R\sin(r/R) & (\kappa = +1) \\ r & (\kappa = 0) \\ R\sinh(r/R) & (\kappa = -1) \end{cases}$$

These metrics can be specified by two quantities, R and κ. κ is called the curvature constant and takes values of +1,0,-1 for positive, flat and negatively curved spaces.

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Does anyone notice something special about these metrics?

They are homogenous and isotropic. They are the only homogenous, isotropic metrics of 3 space.

# PROBLEMS

- Suppose you live on a sphere of radius R, show that if you draw a circle the circumference will be be

$$C = 2\pi R \sin \frac{r}{R}$$

- For the Earth (R=6371 km) how large of a circle must you draw to notice a 1m deviation from Euclidean geometry?

# SPACE-TIME

This section largely follows

*Lecture Notes on General Relativity* by Sean Carroll

that can be found at preposterousuniverse.com/grnotes/

# SPECIAL THEORY OF RELATIVITY

- In 1887, Michelson and Morley showed that there was no relative motion of light, its speed was constant.
- In 1905, Einstein proposed; relativity - that the laws of physics are the same uniform motion, and that the speed of light was the same for all reference frames.
- From this it followed that observers can measure time differently, time and length can be contracted and velocities to not add simply.

# THE LORENTZ FACTOR

- Lorentz and others had already suggested that the physical length of objects might be effected by their motion in an attempt to explain the Michelson-Morley result.
- If the length of an object $L_0$ was contracted to $L=L_0/\gamma$ this could explain why our meter sticks were missing the changing speed of light.
- This amount, $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is called the Lorentz factor.
- They also found that the time coordinate must be effected as well, $t=\gamma t_0$.

# FIRST YEAR RELATIVITY

- The results of special relativity we teach in introductory physics are:
    - Time dilation, $\Delta t = \gamma \Delta t_0$.
    - Lorentz contraction, $L = L_0/\gamma$.
    - Relativistic mass, $m = \gamma m_0$.
    - and $E = mc^2$.
- These results can be derived from writing the laws of motion in a four dimensional space called space-time.

# SPACE-TIME

- Special relativity showed that we do not live in a 3 dimensional world, but instead in a 4 dimensional world of space-time.
- Minkowski (a math professor of Einstein) realized that special relativity could be expressed as a geometry in a 4 dimensional space we now call Minkowski space.
- In this space, points are called events, they have a time value and 3 spatial values $x_{\nu} = (ct,x,y,z)$.
- These vectors are specified by supscript greek letter as opposed to the arrow of 3D vectors.

# FOUR-VECTORS

- All physical quantities can be expressed as four vectors.
- An event: $x_\nu = (ct, x, y, z)$
- The four-momentum: $p_\nu = (E/c, p_x, p_y, p_z)$
- The four-current: $j_\nu = (\rho c, j_x, j_y, j_z)$

# GR NOTATION

- The main thing about GR that is hard is that the notation is new.
- Quantities are expressed as four-vectors or tensors of higher rank. $x_\mu$, $T_{\mu\nu}$, $\varepsilon_{\mu\nu\sigma}$, $R_{\mu\nu\sigma\rho}$.
- Raised or lowered indices *matter*, a lot. They are referred to as covariant or contravariant indices. $x_\mu \neq x^\mu$. Tensors can have mixed variance, $T_{\mu\nu} \neq T_\mu{}^\nu \neq T^{\mu\nu}$.
- The difference in raised (contravariant) and lowered (covariant) indices is similar to the difference between column and row vectors.

$$x_\mu = [ct, x, y, z]$$

$$x^\mu = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

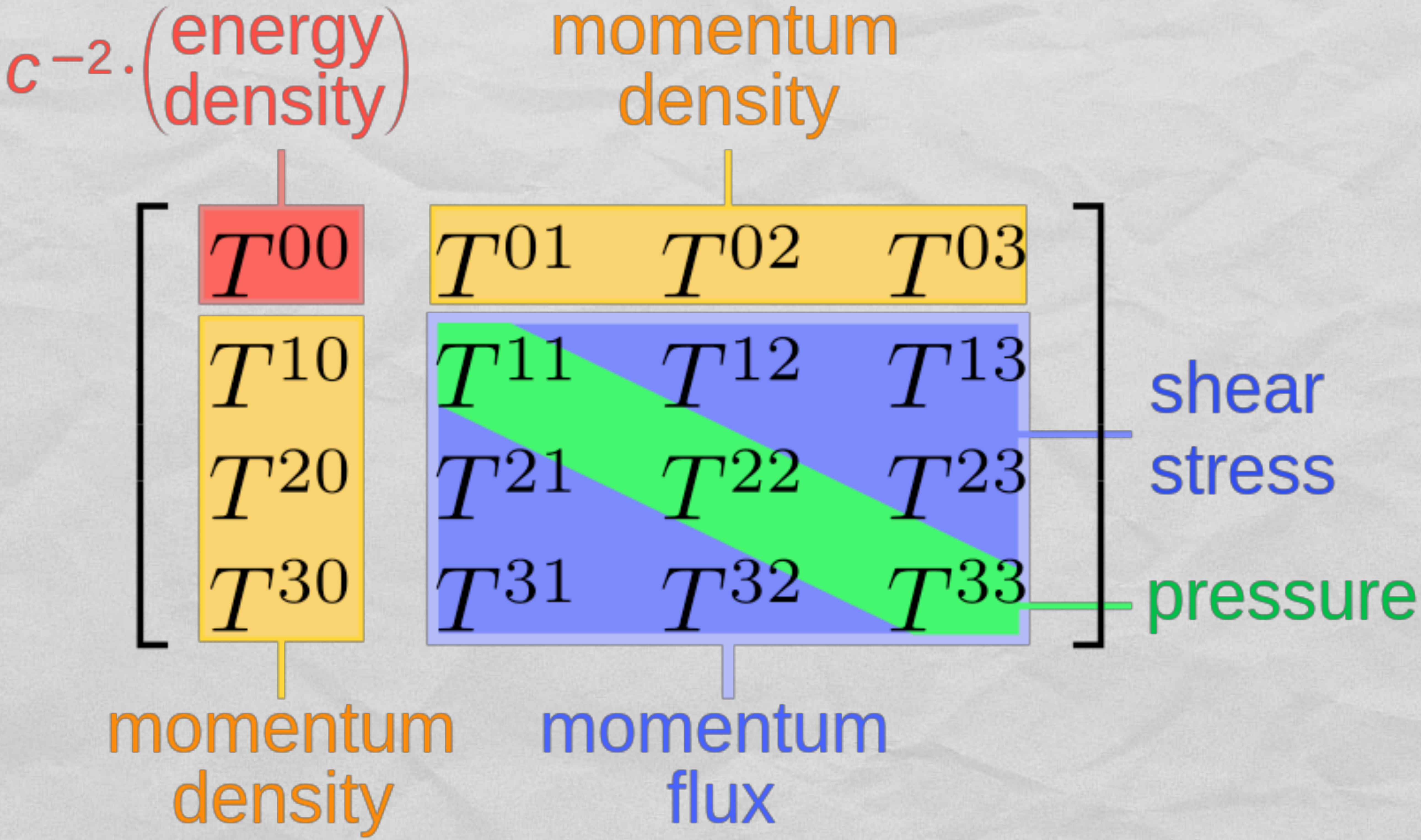
$$n_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The Minkowski metric

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

# THE ENERGY-MOMENTUM TENSOR

- Another important tensor is the energy-momentum or stress-energy tensor, $T^{\mu\nu}$.
- While the 4-vector $p^{\mu}$, describes the momentum and energy of a single particle, for extended systems a 2nd rank tensor is needed.
- This tensor's components will depend on the system in question, but a general class of interest are perfect fluids, which is a fluid that can be completely described by pressure and density.


$c^{-2} \cdot$ (energy density)
momentum density
$T^{00}$ $T^{01}$ $T^{02}$ $T^{03}$
$T^{10}$ $T^{11}$ $T^{12}$ $T^{13}$
$T^{20}$ $T^{21}$ $T^{22}$ $T^{23}$
$T^{30}$ $T^{31}$ $T^{32}$ $T^{33}$
shear stress
pressure
momentum density
momentum flux

# THE ENERGY-MOMENTUM TENSOR

In the rest frame the energy-momentum tensor of a perfect fluid is

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}$$

where ρ is the mass density and P is the pressure. Note that a perfect fluid is isotropic and if uniform is homogenous.

# COVARIANT & CONTRAVARIANT

- Covariant and contravariant refer to how an object changes with a change in basis.
- A covariant object changes in the same way as the basis as is represented with lower indices, $X_\mu$.
- A contravariant object changes opposite to the change of basis and is represented with raised indices, $X^\mu$.

# COVARIANT & CONTRAVARIANT

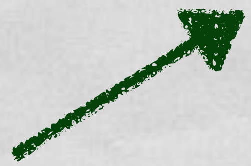
- For example if you have a coordinate system in meters and you divide it by a thousand to get mm, you have to multiply the length of each vector by 1000, that is make the opposite change. So vectors are contravariant.

- On the other hand if you took a gradient in your new coordinate system the gradient would also be divided by 1000. So gradients are covariant.

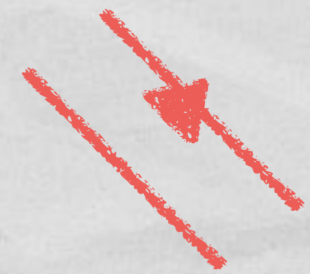
- In Euclidean space the change between a covariant and contravariant quantity is trivial which is why it usually isn't even brought up.

# DIFFERENTIAL FORMS

- A way that more explicitly shows this difference that is far beyond this course are differential forms. However, they have a nice geometrical representation that explains this difference.
- We introduce a new thing called a 1-form, which is covariant. The picture of a 1-form is parallel lines which you should read like a contour map. The closer the lines are the larger the gradient.



vector



1-form

## Stretch the space:

vectors become larger

1-forms become smaller

## The product of a vector and a 1-form is a scalar:

## The product of two vector or 1-forms are 2-vectors or 2-forms.

# EINSTEIN NOTATION

- Einstein summation - if the same letter is used for an index it means that index should be summed over.

$$A_\mu B^\mu = A_0 B^0 + A_1 B^1 + A_2 B^2 + A_3 B^3$$

- To raise or lower an index requires tensor contraction with the metric tensor.

$$A_\nu = g_{\mu\nu} A^\mu$$

- The dot product can thus be defined like so

$$A \cdot B = g_{\mu\nu} A^\mu B^\nu = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3$$

# THE INVERSE METRIC

To raise indices requires the inverse metric

$$A^{\nu} = g^{\mu\nu} A_{\mu}$$

Which is defined by

$$g^{\mu\nu} g_{\nu\sigma} = \delta^{\mu}_{\sigma}$$

Where $\delta$ is the Kronecker delta, which equals 1 if the indices are the same and 0 otherwise.

In flat space-time in Cartesian coordinates the metric and inverse metric are the same, but in general this is not the case.

# SPACE-TIME INTERVAL

The space time interval is just the dot product of the difference between two events

$$s^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

Unlike in a metric space this interval can be positive or negative or zero even though the events are not identical. You can thus have time-like intervals, space-like intervals and null intervals.

$s^2 < 0$ - time like. There can be a causal relationship between the two events. $-s^2/c^2$ is called the proper time

$s^2 > 0$ - space like. The two events are causally disconnected. $s^2$ is called the proper distance.

$s^2 = 0$ - null. A massless particle can only be at two events with a null interval.

# PROPER TIME

Consider the special case of $\Delta x=\Delta y=\Delta z=0$. Then $s^2 = -c^2t^2$. This is just the time that passes for a stationary observer, or what we used to just call time. Now will call it proper time. We will show later that this is the invariant time, the time measure that is the same for all observers.

$$c d\tau = \sqrt{-ds^2}$$

If we want to take a time-like derivative in space-time we want to take it with respect to proper time, otherwise the result will be different for different observers.

# FOUR-VELOCITY

To take derivates with respect to proper time we will need to evaluate dt/dτ.

$$(cd\tau)^2 = (cdt)^2 - d|\vec{x}|^2$$

$$\left(\frac{cd\tau}{cdt}\right)^2 = \left(\frac{cdt}{cdt}\right)^2 - \left(\frac{d\vec{x}}{cdt}\right)^2 = 1 - \frac{v^2}{c^2} = \frac{1}{\gamma(v)^2}$$

$$\implies \frac{dt}{d\tau} = \gamma(v)$$

So now we can define the four-velocity as the derivative of an event with respect to proper time.

$$v^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \gamma(v)\,[c, \vec{v}]$$

# LORENTZ TRANSFORMATIONS

- Relativity states that the laws of physics are the same for all observers, which means that a change of coordinates can't change the physics.
- This implies that physics must be formulated in terms of quantities that are invariant under coordinate transformations.
- For example shifting the coordinates:

$$x^\mu \to x^{\mu'} = x^\mu + a^\mu$$

- Or multiplying by a tensor

$$x^\mu \to x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^\nu$$

# LORENTZ TRANSFORMATIONS

For these transformations to be invariant the path element ds$^2$ must remain unchanged.

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu'\nu'}dx^{\mu'}dx^{\nu'} = \Lambda^{\mu'}{}_\rho \Lambda^{\nu'}{}_\sigma \eta_{\mu'\nu'}dx^\rho dx^\sigma$$

So this will be true for tensors that leave the metric unchanged.

$$\eta_{\rho\sigma} = \Lambda^{\mu'}{}_\rho \Lambda^{\nu'}{}_\sigma \eta_{\mu'\nu'}$$

These transformations are called Lorentz transformations. They form a group called the Lorentz group. Groups are just families of transformations that meet certain requirements.

The Lorentz group consists of ordinary rotations and boosts. If one also includes translations this is called the Poincare group.

# LORENTZ BOOSTS

Ordinary rotations, which just apply to the last 3 indices are Lorentz invariant

$$\Lambda^{\mu'}{}_{\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation in the x-y plane.

A boost is a rotation between the time coordinate and a space coordinate

$$\Lambda^{\mu'}{}_{\nu} = \begin{bmatrix} \cosh\psi & -\sinh\psi & 0 & 0 \\ -\sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# RECOVERING SPECIAL RELATIVITY

A boost is equivalent to a coordinate system with a different velocity (hence the name).

$$t' = t\cosh\psi - x\sinh\psi$$
$$x' = -t\sinh\psi + x\cosh\psi$$

At x'=0 we have

$$t\sinh\psi = x\cosh\psi$$

so $v = \dfrac{x}{t} = \tanh\psi$

if we write the transformation in terms of this *v* we get

$$t' = \gamma(t - vx)$$
$$x' = \gamma(x - vt)$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

we get time dilation and the Lorentz contraction.

# RELATIVISTIC MOMENTUM

Let us now consider the norm of the energy-momentum four vector.

$$p \cdot p = \eta_{\mu\nu} p^\mu p^\nu = \frac{E^2}{c^2} - |\vec{p}|^2$$

But we can also write p in terms of the four-velocity.

$$p \cdot p = m^2 \eta_{\mu\nu} v^\mu v^\nu = m^2 \gamma(v)^2 (c^2 - |\vec{v}|^2) = mc^2$$

which gives us the energy-momentum relation

$$E^2 = (c^2 |\vec{p}|^2) + (mc^2)^2$$

we can always perform a boost so that the particle is at rest. The *rest* energy is then

$$E = mc^2$$

# THE NEXT STEP

- Thus we see that we can recover all of special relativity by considering Lorentz invariants in Minkowski space.
- You may feel that was a lot of work just to get back what we already knew. But you may feel it gives some deeper insight to understand special relativity as a geometric effect.
- Einstein was also skeptical of Minkowski's work at first. However, he found he had to adopt it for GR which can only be formulated as a geometry.

# EM IN SPACE-TIME

- In order to discuss Maxwell's Equations we will have to introduce the 4-gradient.

$$\frac{\partial}{\partial x^\mu} = \left[-\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right]$$

- and the electromagnetic field tensor

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

Then 2 of Maxwell's equations can be expressed as

$$\frac{\partial F^{\mu\nu}}{\partial x^\mu} = \mu_0 J^\nu$$

Homework Problem 2

# GENERAL RELATIVITY

# CURVATURE

- We now want to extend our discussion to curved spaces.
- What this means is that we replace the Minkowski metric, $\eta_{\mu\nu}$, for flat space-time with a general metric $g_{\mu\nu}$.
- This really doesn't change much, but it does change differentiation.

# PARTIAL DERIVATIVES

- In turns out that while partial derivates give invariant answers in flat space, this is not the case in curved spaces.
- Without derivates we would have a hard time doing physics.
- To fix this we create a covariant derivative that fixes this problem.
- All the effects of curvature are basically in this covariant derivative.

# COVARIANT DERIVATIVES

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$

This is just a regular partial derivative plus and extra piece $\Gamma^\nu_{\mu\lambda}$. The extra piece is called a connection coefficient or a Christoffel symbol.

The values of the Christoffel symbol can be derived from the metric.

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left( \frac{\partial}{\partial x^\mu} g_{\nu\rho} + \frac{\partial}{\partial x^\nu} g_{\rho\mu} - \frac{\partial}{\partial x^\rho} g_{\mu\nu} \right)$$

They also obey a bunch of relationships and symmetries that we won't get into here.

# AN EXAMPLE

For illustrative purposes lets work out the Christoffel symbols for a very simple case, polar coordinates. Our metric is

$ds^2 = dr^2 + r^2 d\theta^2$ In matrix notation $\gamma_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$

The inverse metric is then $\gamma^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix}$

Let us use r and θ as the indices instead of 1 and 2.

$$\Gamma^r_{rr} = \frac{1}{2} g^{ri} \left(\frac{\partial}{\partial r} g_{ri} + \frac{\partial}{\partial r} g_{ir} - \frac{\partial}{\partial x^i} g_{rr}\right)$$

$$= \frac{1}{2} g^{rr} \left(\frac{\partial}{\partial r} g_{rr} + \frac{\partial}{\partial r} g_{rr} - \frac{\partial}{\partial r} g_{rr}\right) + \frac{1}{2} g^{r\theta} \left(\frac{\partial}{\partial r} g_{r\theta} + \frac{\partial}{\partial r} g_{\theta r} - \frac{\partial}{\partial \theta} g_{rr}\right)$$

# AN EXAMPLE

$$\Gamma^r_{rr} = \frac{1}{2}g^{ri}(\frac{\partial}{\partial r}g_{ri} + \frac{\partial}{\partial r}g_{ir} - \frac{\partial}{\partial x^i}g_{rr})$$

$$= \frac{1}{2}g^{rr}(\frac{\partial}{\partial r}g_{rr} + \frac{\partial}{\partial r}g_{rr} - \frac{\partial}{\partial r}g_{rr}) + \frac{1}{2}g^{r\theta}(\frac{\partial}{\partial r}g_{r\theta} + \frac{\partial}{\partial r}g_{\theta r} - \frac{\partial}{\partial \theta}g_{rr})$$

$$= \frac{1}{2}(1)(0+0-0) + \frac{1}{2}(0)(0+0-0) = 0$$

Not so interesting

$$\Gamma^r_{\theta\theta} = \frac{1}{2}g^{ri}(\frac{\partial}{\partial \theta}g_{\theta i} + \frac{\partial}{\partial \theta}g_{i\theta} - \frac{\partial}{\partial x^i}g_{\theta\theta})$$

$$= \frac{1}{2}g^{rr}(\frac{\partial}{\partial \theta}g_{\theta r} + \frac{\partial}{\partial \theta}g_{r\theta} - \frac{\partial}{\partial r}g_{\theta\theta})$$

$$= \frac{1}{2}(1)(0+0-2r) = -r$$

The other 6 I'll leave for homework

# EINSTEIN TENSOR

The Riemann curvature tensor expresses the curvature of a space. It measures the noncommutativity of the covariant derivative. It can be expressed in terms of derivatives of the Christoffel symbols.

$$R^{\rho}{}_{\sigma\mu\nu} = \frac{\partial}{\partial x^{\mu}}\Gamma^{\rho}_{\nu\sigma} - \frac{\partial}{\partial x^{\nu}}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

A contraction of the Riemann tensor is known as the Ricci tensor:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$$

A further contraction with the metric gives the Ricci scalar

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu}R_{\mu\nu}$$

Which finally gives us the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu}$$

# AND FINALLY GR

If you are wondering why $G_{\mu\nu}$ the answer is that unlike the Ricci tensor, but like the stress energy tensor it is divergenceless

$$\nabla^\mu G_{\mu\nu} = 0$$

And so now we can get to Einstein's equation, which is to relate space-time curvature $G_{\mu\nu}$ to the stress energy tensor $T_{\mu\nu}$.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Here we write it in its full form with the cosmological constant. Moving that term to the other side of the equation one can think of it as dark energy.

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda g_{\mu\nu}$$

# THAT'S ALL FOLKS!

- You start with a metric, $g_{\mu\nu}$.
- From the metric you get the Christoffel symbols, $\Gamma^{\nu}{}_{\mu\lambda}$.
- From the Christoffel symbols you get the Riemann tensor $R^{\rho}{}_{\sigma\mu\nu}$.
- Which gives you the Ricci tensor, $R_{\mu\nu}$, and scalar, R. Thus the Einstein tensor, $G_{\mu\nu}$.
- And if this satisfies the Einstein equation -is equal to a constant times $T_{\mu\nu}$ - then you have the right metric.

# GR IS HARD

- Note that it is straightforward to go from a metric to the Einstein tensor and the energy-momentum tensor.
- But not in reverse. Starting with an energy-momentum tensor, one has to guess metrics until you find one that works.
- Thus in general GR problems are very hard to solve unless there is a great deal of symmetry in the problem that allows one to guess the metric.

# ROBERTSON-WALKER

- Fortunately, cosmology is one of those cases.
- The cosmological principle: isotropy and homogeneity, means we are looking for a metric that does not depend on spatial location, has no off diagonal terms and x, y and z are the same.

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2/R_0^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]$$

or in terms of conformal time and k instead of κ

$$ds^2 = R(\eta)\left[d\eta^2 - \frac{dr^2}{1-kr^2} - r^2 d\theta^2 - r^2 \sin^2\theta d\theta^2\right]$$

# THE SCALE FACTOR

- This metric is fully specified by $a(t)$, called the scale factor and the curvature $k=\kappa/R_0^2$.
- The expansion factor is normalized such that $a(0) = a_0 = 1.0$, where 0 is today.
- There are some different ways of treating the curvature. One can either have $a(t)$ dimensionless and $k$ a real number with dimensions, or $k$ just -1,0, or 1 and then $a(t)$ has dimensions and can be written as $R(t)$ the radius of the Universe (in a curvature sense).

# AS MATRICES

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2r^2 & 0 \\ 0 & 0 & 0 & a^2r^2\sin^2\theta \end{bmatrix}$$

The metric as a matrix, we see there are no off diagonal terms. The spatial coordinates appear different because of the spherical coordinates.

For flat (k=0) spaces the metric reduces to this and the x,y,z symmetry is obvious.

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{bmatrix}$$